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EIGENVALUE EMBEDDING IN FINITE ELEMENT MODEL UPDATING OF GYROSCOPIC SECOND-ORDER VIBRATING STRUCTURES
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Abstract: The eigenvalue embedding problem addressed in this paper is the one of reassigning a few troublesome eigenvalues of a finite element model to some suitable chosen ones, in such a way that the updated model keeps its symmetry properties and the remaining large number of eigenvalues and eigenvectors of the original model is to remain unchanged. This problem naturally arises in the process of stabilizing large-scale system where dangerous vibrations had been detected, and which can be responsible for undesired phenomena such as resonance. The model matrices are updated using low rank updates that keep their structure regarding symmetry. Algorithm and numerical examples are provided. Numerical experimentation with real-life data, coming from a well-known matrix web repository, have been presented.

Keywords: vibrating, system, gyroscopic, embedding

1. INTRODUCTION

Vibrating structures such as bridges, highways, buildings, automobiles, air and space crafts, and others, are very often modelled by using finite-element methods. These methods generate structured systems of matrix second-order differential equations of the form

$$M\ddot{x} + C\dot{x} + Kx = 0, \tag{1}$$

where the coefficient matrices M, C and K are called, respectively, the mass, damping and stiffness matrices. In most applications, these matrices have very special exploitable properties such as the symmetry, positive definiteness, sparsity and others. The matrix M is often symmetric positive definite and denoted by $M > 0$; and K is usually symmetric positive semi-definite, denoted by $K \geq 0$. The damping matrix C is hard to determine in practice; however, very often, it is regarded either as symmetric ($C^T = C$) or skew-symmetric (gyroscopic) ($C^T = -C$), and there are important applications for both cases in literature.

It is critical and very important that these properties are preserved while solving a vibration problem or updating a finite element model to achieve important design objectives.

The classical approach [18] is to use separation of variables, accounting for a solution $x(t) = ye^{\lambda t}$ to (1), where y is a constant vector. This leads to the quadratic matrix eigenvalue problem

$$F(\lambda_k)y_k = 0, \ k = 1, 2, \ldots, 2n,$$

where

$$F(\lambda) = \lambda^2 M + \lambda C + K \tag{2}$$

is the so-called associated quadratic matrix pencil. The quantities (λ_k, y_k), $k = 1, \ldots, 2n$ are the eigenpairs of the pencil (2).

It is well-known [18] that the dynamical behavior of a vibrating system, which can show undesired phenomena such as instability and resonance, is determined by their natural frequencies and corresponding mode shapes, that is, the eigenvalues and eigenvectors of the pencil $F(\lambda)$. It is desirable that such behaviors are altered by making minimal changes in the system and keeping the structural properties invariant, as much as possible. Realistically, while dealing with a large system, it is often found in practice that only a small number of eigenvalues are “troublesome”. Thus, it makes sense to reassign to suitable locations, chosen by the designer, only these troublesome eigenvalues, while keeping the remaining large number of eigenvalues unchanged.

In several recent papers [4–6, 8, 10] numerically effective methods have been developed for both partial pole-placement and eigenstructure assignment problems. These methods are designed directly in matrix second order setting without resorting to first order transformations and without requiring complete knowledge of the spectrum of the pencil $F(\lambda)$, as needed by the IMSC approach [18]. Although they satisfy control design requirements and are practical for control applications, unfortunately, they are not capable of preserving the symmetry of the original model.

In a recent paper [2], a novel *symmetry preserving partial spectrum assignment method* for vibrating system (1) was proposed. Specifically, the following problem was solved:

Given system matrices M, C, and K of the model (1) with $M = M^T > 0$ and $K = K^T > 0$, and C is symmetric, a part of the spectrum $\{\lambda_1, ..., \lambda_r\}$, $r \leq 2n$ of $F(\lambda)$, and a set of r complex numbers $\{\mu_1, \ldots, \mu_r\}$; both the sets $\{\lambda_1, \ldots, \lambda_r\}$ and $\{\mu_1, \ldots, \mu_r\}$ closed under complex conjugation, find real symmetric matrices M_{new}, K_{new} and C_{new} such that the spectrum of $F_{new} = \lambda^2 M_{new} + \lambda C_{new} + K_{new}$ is $\{\mu_1, \ldots, \mu_r, \lambda_{r+1}, \ldots, \lambda_{2n}\}$.

To distinguish the problem above from the partial pole-placement problem in control theory, this problem was called "**Eigenvalue Embedding**" Problem (EEP).

In this paper, we address the solution of the EEP for a gyroscopic second order system, meaning that its *mass* ,*stiffness* and *damping* matrices satisfy $M = M^T > 0, K = K^T > 0$ and $C^T = -C$, respectively.

Our major contributions to EEP in the gyroscopic setup, presented in this paper are as follows:

(i) An algorithm and associated theories are developed, using low-rank updates that are symmetric for M and K, but skew-symmetric for C.

(ii) The accuracy of the algorithm is demonstrated by using both illustrative, and a real-life example with simulated data from the Boeing Company.

(iii) A complete characterization of the eigenvectors of the updated model is also given. Following the strategies in [4] , a set of orthogonality relations between eigenvectors of gyroscopic second order systems is derived. From that, we can show by mathematical proofs that the eigenvectors corresponding to the eigenvalues which are not reassigned also remain invariant.

The last property is highly significant from practical applications view points. It says that certain important physical properties of the system are completely preserved by updating.

The solution proposed in this paper for EEP can be considered as a partial but meaningful solution to the Finite Element Model Updating problem.

2. ORTHOGONALITY RELATIONS FOR THE GYROSCOPIC CASE

We first observe that it can be shown, under the hypothesis above for M, C and K, that the eigenvalues λ of the matrix pencil defined in (2) are always purely imaginary, and therefore we can write $\lambda = i\beta$, where $\beta \in \mathbb{R}$ is the so-called natural frequency. Let x be the corresponding unitary eigenvector (it usually has complex components). Since $\bar{\lambda} = -\lambda$ is also an eigenvalue of the pencil (2), it gives a natural frequency $-\beta$ and a unitary eigenvector $\bar{x}$. Consider the eigenspace decomposition

$$MX\Lambda^2 + CX\Lambda + KX = 0 \tag{3}$$

where Λ is a block diagonal matrix containing 2×2 blocks, corresponding to every purely imaginary pair of eigenvalues, on its diagonal, while X is a block row matrix containing $n \times 2$ blocks containing information on the corresponding eigenvectors.

This representation can be done in two ways: consider a pair of purely imaginary eigenvalues $\pm\beta_i i$ and corresponding eigenvectors $y_{ir} \pm y_{ii}$, where y_{ir} and y_{ii} are linearly independent real vectors.

Complex representation: in this case, we use

$$\Lambda_i = \begin{bmatrix} \beta_i i & 0 \\ 0 & -\beta_i i \end{bmatrix}, \; X_i = \begin{bmatrix} y_{ir} + iy_{ii} & y_{ir} - iy_{ii} \end{bmatrix}$$

and therefore $\overline{\Lambda} = -\Lambda, \Lambda^T = \Lambda$.

Real representation: in this case, we use

$$\Lambda_i = \begin{bmatrix} 0 & \beta_i \\ -\beta_i & 0 \end{bmatrix}, \; X_i = \begin{bmatrix} y_{ir} & y_{ii} \end{bmatrix}$$

and therefore $\overline{\Lambda} = \Lambda, \Lambda^T = -\Lambda$.

Therefore, no matter the representation, we are sure to have $(\overline{\Lambda})^T = -\Lambda$.

Theorem 2.1 (Orthogonality Relations) *If all the eigenvalues of the pencil (2) are nonzero, then*

(i) the matrix

$$D_1 = \overline{(X\Lambda)}^T M(X\Lambda) + \overline{X}^T KX$$

is block diagonal, where the size of each diagonal block corresponds to the multiplicity of the corresponding eigenvalue.

(ii) the matrix

$$D_2 = (X\Lambda)^T M(X\Lambda) + X^T KX.$$

is block diagonal, where the size of each diagonal block corresponds to at most twice the multiplicity of the corresponding eigenvalue.

Proof of (i): First, we left-multiply (3) by $\overline{(X\Lambda)}^T$ to get

$$-\overline{(X\Lambda)}^T C(X\Lambda) = \overline{(X\Lambda)}^T M(X\Lambda)\Lambda + \overline{(X\Lambda)}^T KX. \tag{4}$$

On another hand, we take the conjugate transpose of (3) and right-multiply it by $(X\Lambda)$ to get

$$-\overline{(X\Lambda)}^T C^T(X\Lambda) = \overline{\Lambda}^T \, \overline{(X\Lambda)}^T M(X\Lambda) + \overline{X}^T KX\Lambda. \tag{5}$$

Now, combining (4) and (5) and using $C^T = -C$ gives us

$$\begin{aligned} \overline{(X\Lambda)}^T M(X\Lambda)\Lambda + \overline{(X\Lambda)}^T KX = \\ -\left[\overline{\Lambda}^T \, \overline{(X\Lambda)}^T M(X\Lambda) + \overline{X}^T KX\Lambda\right] \end{aligned} \tag{6}$$

and using $\overline{\Lambda}^T = -\Lambda$ gives us

$$\left[\overline{(X\Lambda)}^T M(X\Lambda) + \overline{X}^T KX\right]\Lambda = \Lambda\left[\overline{(X\Lambda)}^T M(X\Lambda) + \overline{X}^T KX\right], \tag{7}$$

that is, $D_1\Lambda = \Lambda D_1$, meaning that D_1 commutes with Λ.

If we assume that the eigenvalues along the diagonal of Λ are placed so that purely imaginary eigenvalues of arbitrary multiplicity all group together, then we conclude that D_1 must be a block diagonal matrix, where the size r of each block equals to the multiplicity of the corresponding eigenvalue.

In the particular case that all the eigenvalues of (1) are distinct, D_1 must be a diagonal matrix.

Proof of (ii): First, we left-multiply (3) by $(X\Lambda)^T$ to get

$$-(X\Lambda)^T C(X\Lambda) = (X\Lambda)^T M(X\Lambda)\Lambda + (X\Lambda)^T KX. \quad (8)$$

On another hand, we take the transpose of (3) and right-multiply it by $(X\Lambda)$ to get

$$-(X\Lambda)^T C^T(X\Lambda) = \Lambda^T(X\Lambda)^T M(X\Lambda) + X^T KX\Lambda. \quad (9)$$

Now, combining (8) and (9) and using $C^T = -C$ gives us

$$\begin{array}{c}(X\Lambda)^T M(X\Lambda)\Lambda + (X\Lambda)^T KX = \\ -\left[\Lambda^T(X\Lambda)^T M(X\Lambda) + X^T KX\Lambda\right].\end{array} \quad (10)$$

Now two cases must be considered:

(a) If complex representation is being used, then $\Lambda^T = \Lambda$, and we conclude

$$\begin{array}{c}\left[(X\Lambda)^T M(X\Lambda) + X^T KX\right]\Lambda = \\ -\Lambda\left[(X\Lambda)^T M(X\Lambda) + X^T KX\right]\end{array} \quad (11)$$

meaning that $D_2\Lambda = -\Lambda D_2$, and then D_2 is block diagonal, having $2r \times 2r$ skew-symmetric blocks on its diagonal, corresponding to purely imaginary eigenvalue pairs of multiplicity r. In particular, if all the eigenvalues of (1) are nonzero and distinct, then D_2 is a diagonal block matrix with 2×2 blocks.

(b) If real representation is being used, then $\Lambda^T = -\Lambda$, and we conclude

$$\begin{array}{c}\left[(X\Lambda)^T M(X\Lambda) + X^T KX\right]\Lambda = \\ \Lambda\left[(X\Lambda)^T M(X\Lambda) + X^T KX\right]\end{array} \quad (12)$$

meaning that $D_2\Lambda = \Lambda D_2$, and then D_2 commutes with Λ, which is a block diagonal matrix having skew-symmetric 2×2 blocks on its diagonal. As a consequence, it is straightforward to show that D_2 is also block diagonal, with diagonal blocks with size equal to the multiplycity of the corresponding eigenvalue.

$\diamond$

Corollary 2.1 *Let (Λ_1, X_1) e (Λ_2, X_2) be any eigenpairs coming from disjoint self-conjugated sets of eigenvalues of the pencil (2). Then*

$$(Y_1\Lambda_1)^T MY_2\Lambda_2 + Y_1^T KY_2 = 0. \quad (13)$$

Proof: Let

$$\Lambda = \begin{bmatrix} \Lambda_1 & & \\ & \Lambda_2 & \\ & & \Lambda_3 \end{bmatrix}, X = \begin{bmatrix} X_1 & X_2 & X_3 \end{bmatrix} \quad (14)$$

where (Λ_3, X_3) contains all the other eigenpairs. Now substituting (14) in the result of part (ii) of Theorem 2.1 gives

$$\begin{array}{l}\begin{bmatrix} (Y_1\Lambda_1)^T \\ (Y_2\Lambda_2)^T \\ (Y_3\Lambda_3)^T \end{bmatrix} M \begin{bmatrix} Y_1\Lambda_1 & Y_2\Lambda_2 & Y_3\Lambda_3 \end{bmatrix} + \\ \qquad \begin{bmatrix} Y_1^T \\ Y_2^T \\ Y_3^T \end{bmatrix} K \begin{bmatrix} Y_1 & Y_2 & Y_3 \end{bmatrix} = \\ \qquad = D_2 = \begin{bmatrix} P_{11} & 0 & 0 \\ 0 & P_{22} & 0 \\ 0 & 0 & P_{33} \end{bmatrix}\end{array} \quad (15)$$

where D_2 has block diagonal structure and P_{11} has the same size as Λ_1, as a consequence of the hypothesis.

Now equating the blocks for the first row, and second column, gives the desired result.

3. EMBEDDING OF A PAIR OF PURELY IMAGINARY EIGENVALUES

The strategy applied here derives from the one presented in [2].

The main goal is to show how to compute updated symmetric matrices M_{new}, K_{new} and C_{new}, such that a pair of purely imaginary eigenvalues, $\lambda = i\beta_1$ and $\overline{\lambda} = -i\beta_1$ is assigned to the spectrum of the corresponding $F_{new}(\lambda)$, while the other eigenvalues of $F_{new}(\lambda)$ remain the same as those of $F(\lambda)$. For simplicity, a matrix pair (Λ, Y) satisfying

$$MY\Lambda^2 + CY\Lambda + KY = 0 \quad (16)$$

will be called an eigenpair of $F(\lambda)$. The notation spec (T) stands for spectrum of the matrix T.

Let $(i\beta_1, y_1)$ be an isolated eigenpair of $F(\lambda)$, with $\beta_1 \in \mathbb{R}$, $\beta_1 \neq 0$, and $y_1 = y_{1r} + iy_{1i}$, y_{1r}, $y_{1i} \in \mathbb{R}^n$. Suppose that y_{1r} and y_{1i} are linearly independent, then y_1 and $\bar{y}_1$ are linearly independent, and $(-i\beta_1, \bar{y}_1)$ is also an eigenpair of $F(\lambda)$. Since $(i\beta_1, y_1)$ is an eigenpair of $F(\lambda)$, we have

$$MZ_1\underline{\Lambda}_1^2 + CZ_1\underline{\Lambda}_1 + KZ_1 = 0, \quad (17)$$

where

$$\underline{\Lambda}_1 = \begin{bmatrix} 0 & \beta_1 \\ -\beta_1 & 0 \end{bmatrix} \text{ and } Z_1 = \begin{bmatrix} y_{1r} & y_{1i} \end{bmatrix}. \quad (18)$$

Thus, $(\underline{\Lambda}_1, Z_1)$ is an eigenpair of $F(\lambda)$. Since K is positive definite, $\Pi_1 = Z_1^\top KZ_1$ is also positive definite. Thus there exists an orthogonal matrix $S_1 \in \mathbb{R}^{2\times 2}$, and a positive diagonal matrix

$$D_1 = \begin{bmatrix} d_1 & 0 \\ 0 & d_2 \end{bmatrix},$$

such that

$$\Pi_1 = S_1 D_1 D_1 S_1^\top. \quad (19)$$

Therefore, the definitions

$$Y_1 = Z_1 S_1 D_1^{-1}, \quad (20)$$

$$\Lambda_1 = D_1 S_1^\top \underline{\Lambda}_1 S_1 D_1^{-1} \quad (21)$$

clearly imply

$$Y_1^\top K Y_1 = I_2$$

$$\Lambda_1 = \begin{bmatrix} d_1 & 0 \\ 0 & d_2 \end{bmatrix} \begin{bmatrix} 0 & \beta_1 \\ -\beta_1 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{d_1} & 0 \\ 0 & \frac{1}{d_2} \end{bmatrix} = \begin{bmatrix} 0 & \beta_1/d \\ -d\beta_1 & 0 \end{bmatrix},$$

where $d = d_1/d_2$.

Theorem 3.1 *Given a nonzero real number σ_1, there is a real diagonal matrix E_M such that $\mu_1 = i\sigma_1$ and $\overline{\mu}_1 = -i\sigma_1$ are eigenvalues of the matrix par $(\Lambda_1\Lambda_1^T - E_M, \Lambda_1^T + E_M\Theta_1\Lambda_1^T)$, where $\Theta_1 = Y_1^T M Y_1$ and Λ_1, Y_1 are given by (20) and (21), respectively.*

Proof: Let

$$\Theta_1 = Y_1^T M Y_1 = \begin{bmatrix} \theta_{11} & \theta_{12} \\ \theta_{12} & \theta_{22} \end{bmatrix} \text{ and } E_M = \begin{bmatrix} \xi & 0 \\ 0 & \eta \end{bmatrix} \in \mathbb{R}^{2\times 2}, \tag{22}$$

where ξ, η are two unknowns. By expanding $det(i\sigma_1 B - A)$, where $A = \Lambda_1\Lambda_1^T - E_M, B = \Lambda_1^T + E_M\Theta_1\Lambda_1^T$, and matching real and imaginary parts to zero, we conclude that ξ, η satisfy a system of two real two degree polynomials

$$\begin{cases} p_0 + p_1\xi + p_2\eta + p_3\xi\eta = 0 \\ q_0 + q_1\xi + q_2\eta + q_3\xi\eta = 0 \end{cases} \tag{23}$$

where

$$\begin{array}{ll} p_0 = 0 & p_1 = \beta_1^2 \\ p_2 = -\beta_1^2 d & p_3 = 1 - d^2 \\ q_0 = \beta_1^2(\beta_1^2 - \sigma_1^2) & q_1 = -\beta_1^2(1/d^2 + \theta_{11}\sigma_1^2) \\ q_2 = -\beta_1^2(d^2 + \theta_{22}\sigma_1^2) & q_3 = 1 - \sigma_1^2\beta_1^2\det(\Theta_1) \end{array} \tag{24}$$

Theorem 3.2 (Embedding Purely Imaginary Eigenvalues) *Let Y_1 and Λ_1 be the same as those defined in (20) and (21). Let E_M be the same as in Theorem 3.1. Define*

$$\begin{array}{l} M_{new} = M + MY_1E_MY_1^TM \\ C_{new} = C + MY_1E_CY_1^TK - KY_1E_C^TY_1^TM \\ K_{new} = K - KY_1E_KY_1^TK \end{array} \tag{25}$$

where

$$E_K = \Lambda_1^{-1}E_M\Lambda_1^{-\top}, \quad E_C = E_M\Lambda_1^{-\top}. \tag{26}$$

Then the real symmetric pencil $F_{new}(\lambda) = \lambda^2 M_{new} + \lambda C_{new} + K_{new}$, has the following properties

(i) The eigenvalues of the matrix pencil $F_{new}(\lambda)$ are the same as those of $F(\lambda)$ except that the purely imaginary eigenvalues $\lambda_1 = i\beta_1, \bar{\lambda}_1 = -i\beta_1$ of $F(\lambda)$ are replaced by the purely imaginary numbers $\mu_1 = i\sigma_1, \bar{\mu}_1 = -i\sigma_1$

(ii) The eigenvectors associated with the other eigenvalues remain the same as those of the original pencil.

(iii) After the updating, the eigenvector matrix Z_1, associated with λ_1 and $\overline{\lambda}_1$ and which satisfies (17), , gets changed into a matrix W_1 given by $W_1 = Y_1U_1$, where Y_1 is defined by (20) and for some matrix $U_1 \in \mathbb{R}^{2\times 2}$.

Proof of (i). From (17) and the definitions of Y_1 and Λ_1, we see that (Λ_1, Y_1) is an eigenpair of $F(\lambda)$ and therefore,

$$MY_1\Lambda_1^2 + CY_1\Lambda_1 + KY_1 = 0.$$

Now, letting $\Lambda = \lambda I_2$, we have

$$\begin{aligned} F(\lambda)Y_1 &= (\lambda^2 M + \lambda C + K)Y_1 \\ &= MY_1\Lambda^2 + CY_1\Lambda - MY_1\Lambda_1^2 - CY_1\Lambda_1 = \\ &= (MY_1(\Lambda + \Lambda_1) + CY_1)(\Lambda - \Lambda_1). \end{aligned} \tag{27}$$

From (25)-(27), we obtain

$$\begin{aligned} F_{new}(\lambda) &= \lambda^2 M_{new} + \lambda C_{new} + K_{new} \\ &= F(\lambda) + \lambda^2 MY_1E_MY_1^\top M - \lambda KY_1E_C^\top Y_1^\top M \\ &\quad + \lambda MY_1E_CY_1^\top K - KY_1E_KY_1^\top K \\ &= F(\lambda) + \lambda(\lambda MY_1\Lambda_1 - KY_1)E_K\Lambda_1^TY_1^TM + \\ &\quad + (\lambda MY_1\Lambda_1 - KY_1)E_KY_1^TK \\ &= F(\lambda) + (\lambda MY_1\Lambda_1 - KY_1)E_K(Y_1^TK + \lambda\Lambda_1^TY_1^TM) = \\ &F(\lambda) + (MY_1(\Lambda + \Lambda_1) + CY_1)\Lambda_1E_K(Y_1^TK + \lambda\Lambda_1^TY_1^TM) \\ &= F(\lambda) + F(\lambda)Y_1(\Lambda - \Lambda_1)^{-1}\Lambda_1E_K(Y_1^TK + \lambda\Lambda_1^TY_1^TM) \\ &= F(\lambda)\left[I_n + Y_1(\Lambda - \Lambda_1)^{-1}\Lambda_1E_K(Y_1^TK + \lambda\Lambda_1^TY_1^TM)\right] \end{aligned}$$

and therefore

$$\begin{aligned} &\det(F_{new}(\lambda)) = \det(F(\lambda)) \times \\ &\det\left[I_n + Y_1(\Lambda - \Lambda_1)^{-1}\Lambda_1E_K(Y_1^TK + \lambda\Lambda_1^TY_1^TM)\right] \\ &= \det(F(\lambda))\det\left[I_2 + (\Lambda - \Lambda_1)^{-1}\Lambda_1E_K(Y_1^TKY_1 + \right. \\ &\left. \lambda\Lambda_1^TY_1^TMY_1)\right] \\ &= \det(F(\lambda))\det\left[(\Lambda - \Lambda_1)^{-1}\right] \times \\ &\det\left[\Lambda - \Lambda_1 + \Lambda_1E_K(I_2 + \lambda\Lambda_1^T\Theta_1)\right] \\ &= \frac{\det(F(\lambda))\det\left[\lambda(I_2 + \Lambda_1E_K\Lambda_1^T\Theta_1) - \Lambda_1 + \Lambda_1E_K\right]}{(\lambda - i\beta_1)(\lambda + i\beta_1)} \\ &= \frac{\det(F(\lambda))\det\left[\lambda(I_2 + E_M\Theta_1) - \Lambda_1 + E_M\Lambda_1^{-T}\right]}{(\lambda - i\beta_1)(\lambda + i\beta_1)} \end{aligned}$$

and finally, since $\det(\Lambda_1) \neq 0$, provided

$$\begin{array}{l} \det\left[\lambda(I_2 + E_M\Theta_1) - \Lambda_1 + E_M\Lambda_1^{-T}\right] = \\ \det\left[\lambda(\Lambda_1^T + E_M\Theta_1\Lambda_1^T) - (\Lambda_1\Lambda_1^T - E_M)\right] = 0 \end{array} \tag{28}$$

part (i) follows.

Proof of (ii). Let $\lambda_2 = i\beta_2$ be another eigenvalue (distinct from λ_1) and let $y_2 = y_{2r} + iy_{2i}$ be the corresponding eigenvector. Define Y_2 and Λ_2 in the same way as Y_1 and Λ_1 have been defined. Then (Λ_1, Y_1) and (Λ_2, Y_2) are eigenpairs of $F(\lambda)$, with $Y_1^\top KY_1 = I_2$ and $Y_2^\top KY_2 = I_2$. Thus

$$\begin{aligned} &M_{new}Y_2\Lambda_2^2 + C_{new}Y_2\Lambda_2 + K_{new}Y_2 = \\ &(M + MY_1E_MY_1^TM)Y_2\Lambda_2^2 + (K - KY_1E_KY_1^TK)Y_2 + \\ &(C + MY_1E_CY_1^TK - KY_1E_C^TY_1^TM)Y_2\Lambda_2 \\ &= MY_1E_MY_1^TMY_2\Lambda_2^2 + MY_1E_CY_1^TKY_2\Lambda_2 \\ &-KY_1E_C^TY_1MY_2\Lambda_2 - KY_1E_KY_1^TKY_2 \\ &= MY_1\Lambda_1E_K(\Lambda_1^TY_1^TMY_2\Lambda_2 + Y_1^TKY_2)\times \\ &\Lambda_2 - KY_1E_K(\Lambda_1^TY_1^TMY_2\Lambda_2 + Y_1^TKY_2) = 0 \end{aligned} \tag{29}$$

because the common term in parenthesis vanishes by Corollary 2.1.

Proof of (iii). By Theorem 3.1, there exists a nonsingular matrix $V_1 \in \mathbb{R}^{2\times 2}$ such that

$$(\Lambda_1^T + E_M\Theta_1\Lambda_1^T)V_1\Sigma_1 = (\Lambda_1\Lambda_1^T - E_M)V_1 \qquad (30)$$

and therefore

$$(I_2 + E_M\Theta_1)U_1\Sigma_1 = \Lambda_1(I_2 - E_K)U_1 \qquad (31)$$

where $U_1 = \Lambda_1^T V_1$. Now writing $W_1 = Y_1U_1$ gives

$$\begin{aligned}
&M_{new}W_1\Sigma_1^2 + C_{new}W_1\Sigma_1 + K_{new}W_1\\
&= (M + MY_1E_MY_1^TM)Y_1U_1\Sigma_1^2+\\
&+(C + MY_1E_CY_1^TK - KY_1E_C^TY_1^TM)Y_1U_1\Sigma_1\\
&+(K - KY_1E_KY_1^TK)Y_1U_1\\
&= MY_1(U_1\Sigma_1^2 + E_M\Theta_1U_1\Sigma_1^2 + E_CU_1\Sigma_1) + CY_1U_1\Sigma_1+\\
&KY_1(-E_C^T\Theta_1U_1\Sigma_1 + U_1 - E_KU_1)\\
&= MY_1(U_1\Sigma_1^2 + E_M\Theta_1U_1\Sigma_1^2 + E_CU_1\Sigma_1) + CY_1U_1\Sigma_1+\\
&+(-MY_1\Lambda_1^2 - CY_1\Lambda_1)(-E_K^T\Lambda_1^T\Theta_1U_1\Sigma_1 + U_1 - E_KU_1)\\
&= MY_1(U_1\Sigma_1^2 + E_M\Theta_1U_1\Sigma_1^2 + E_CU_1\Sigma_1)\\
&-MY_1\Lambda_1^2(-E_K^T\Lambda_1^T\Theta_1U_1\Sigma_1 + U_1 - E_KU_1)+\\
&+CY_1\left[(I_2 + E_M\Theta_1)U_1\Sigma_1 - \Lambda_1(I_2 - E_K)U_1\right]\\
&= MY_1(U_1\Sigma_1^2 + E_M\Theta_1U_1\Sigma_1^2 + \Lambda_1E_KU_1\Sigma_1-\\
&\Lambda_1^2(-E_K^T\Lambda_1^T\Theta_1U_1\Sigma_1 - (I_2 - E_K)U_1))+\\
&-MY_1\Lambda_1U_1\Sigma_1 + MY_1\Lambda_1U_1\Sigma_1\\
&= MY_1\left[(I_2 + E_M\Theta_1)U_1\Sigma_1 - \Lambda_1(I_2 - E_K)U_1\right]\Sigma_1+\\
&+MY_1\Lambda_1((E_M\Theta_1U_1\Sigma_1 - \Lambda_1(I_2 - E_K)U_1 + U_1\Sigma_1)\\
&= MY_1\Lambda_1\left[(I_2 + E_M\Theta_1)U_1\Sigma_1 - \Lambda_1(I_2 - E_K)U_1\right] = 0
\end{aligned} \qquad (32)$$

since all the terms in brackets vanish by (31).

4. ALGORITHM AND NUMERICAL RESULTS

In this section, we present an algorithm to accomplish the proposed Eigenvalue Embedding strategy, as well as to illustrate the efficiency and reliability of the proposed method by means of a numerical example.

Algorithm 4.1 *Embedding of a Pair of Purely Imaginary Eigenvalues in a Gyroscopic Model*

Input: System matrices M, C and K such that M, and K are symmetric (K is positive definite), and C is skew-symmetric; quantities β_1 and y_1 such that $(i\beta_1, y_1)$ is an isolated eigenpair of $F(\lambda) = \lambda^2M + \lambda C + K$. real nonzero number σ_1.

Step 1. Form matrices

$$\underline{\Lambda}_1 = \begin{bmatrix} 0 & \beta_1 \\ -\beta_1 & 0 \end{bmatrix},\ Z_1 = \begin{bmatrix} y_{1r} & y_{1i} \end{bmatrix}$$

where y_{1r} and y_{1i} are the real an imaginary parts of vector y_1, respectively.

Step 2. Form matrices Λ_1, Y_1 and Θ_1 according to (19)-(22).

Step 3. Compute ξ and η according to Theorem 3.1.

Step 4. Form matrix E_M in (22) and compute matrices E_C and E_K given by

$$E_K = \Lambda_1^{-1}E_M\Lambda_1^{-\top}, \quad E_C = E_M\Lambda_1^{-\top}.$$

Step 5. Compute the rank-2 updates

$$\begin{aligned}
M_{new} &= M + MY_1E_MY_1^TM\\
C_{new} &= C + MY_1E_CY_1^TK - KY_1E_C^TY_1^TM\\
K_{new} &= K - KY_1E_KY_1^TK
\end{aligned}$$

Numerical Example 1. Consider the vibrating system whose system matrices are

$$M = \begin{bmatrix} 8.894500 & 0.591800 & -1.157400 & 4.837800\\ 0.591800 & 16.380400 & -2.453300 & -1.467400\\ -1.157400 & -2.453300 & 16.544200 & -4.413700\\ 4.837800 & -1.467400 & -4.413700 & 7.411900 \end{bmatrix}$$

$$C = \begin{bmatrix} 0. & -4.471900 & 1.609200 & -2.210000\\ 4.471900 & 0. & -0.023400 & -2.422800\\ -1.609200 & 0.023400 & 0. & -0.452700\\ 2.210000 & 2.422800 & 0.452700 & 0. \end{bmatrix}$$

$$K = \begin{bmatrix} 19.435200 & -6.691200 & 0.268900 & -1.811600\\ -6.691200 & 17.174300 & -0.553300 & 0.338900\\ 0.268900 & -0.553300 & 12.054000 & -4.433000\\ -1.811600 & 0.338900 & -4.433000 & 14.356500 \end{bmatrix}$$

and whose real eigensystem decomposition [1] is given by (Step 1)

$$\underline{\Lambda}_1 = \begin{bmatrix} 0. & 0.792217\\ -0.792217 & 0. \end{bmatrix}$$

$$\underline{\Lambda}_2 = \begin{bmatrix} 0. & .9037 & 0. & 0. & 0. & 0.\\ -.9037 & 0. & 0. & 0. & 0. & 0.\\ 0. & 0. & 0. & 1.1275 & 0. & 0.\\ 0. & 0. & -1.1275 & 0. & 0. & 0.\\ 0. & 0. & 0. & 0. & 0. & 2.8395\\ 0. & 0. & 0. & 0. & -2.8395 & 0. \end{bmatrix}$$

$$Z_1 = \begin{bmatrix} -0.353009 & 0.507066\\ -0.115653 & 0.846237\\ -0.062387 & -0.998052\\ 0.049946 & 0.061286 \end{bmatrix}$$

$$Z_2 = \begin{bmatrix} .13466 & -.49704 & -.07346 & -.55556 & .06773 & -.83266\\ .13162 & -.73825 & -.03368 & .44198 & .15789 & .22900\\ -.16812 & -.98577 & -.20703 & -.09157 & .02878 & .22317\\ -.3105 & -.36543 & -.09940 & -.99505 & .07525 & .99717 \end{bmatrix}.$$

Computation in Step 2 gives

$$D_1 = \begin{bmatrix} 4.9867 & 0.\\ 0. & 1.4679 \end{bmatrix},\ S_1 = \begin{bmatrix} -0.0741 & 0.9972\\ 0.9972 & 0.0741 \end{bmatrix}$$

$$Y_1 = \begin{bmatrix} 0.106653 & -0.214219\\ 0.170953 & -0.035831\\ -0.198666 & -0.092795\\ 0.011514 & 0.037028 \end{bmatrix}$$

$$\Lambda_1 = \begin{bmatrix} 0. & -2.6913\\ 0.2332 & 0. \end{bmatrix},\ \Theta_1 = \begin{bmatrix} 1.4974 & 0.\\ 0. & 0.4861 \end{bmatrix}$$

[1] We remark that $\underline{\Lambda}_2$ and Z_2 are computed just for the sake of checking the invariance of the non-updated spectrum, at the end of the example. They are never needed by the proposed algorithm.

After choosing $\sigma_1 = 0.67$, computation is Steps 3 and 4 give

$$E_M = 10^{-1}\begin{bmatrix} 0.509308 & 0. \\ 0. & 0.119763 \end{bmatrix}$$
$$E_C = 10^{-1}\begin{bmatrix} 0. & -0.189240 \\ 0.513573 & 0. \end{bmatrix}$$
$$E_K = \begin{bmatrix} 0.220232 & 0. \\ 0. & 0.007031 \end{bmatrix}$$

Computation in Step 5 gives

$$M_{new} = \begin{bmatrix} 9.017542 & 0.829167 & -1.394559 & 4.927162 \\ 0.829167 & 16.949983 & -3.103368 & -1.257067 \\ -1.394559 & -3.103368 & 17.333347 & -4.651364 \\ 4.927162 & -1.257067 & -4.651364 & 7.489692 \end{bmatrix}$$
$$C_{new} = \begin{bmatrix} 0. & -4.920589 & 2.208825 & -2.400146 \\ 4.920589 & 0. & 0.228364 & -2.474837 \\ -2.208825 & -0.228364 & 0. & -0.488924 \\ 2.400146 & 2.474837 & 0.488924 & 0. \end{bmatrix}$$
$$K_{new} = \begin{bmatrix} 19.160943 & -7.106027 & 0.704376 & -1.945778 \\ -7.106027 & 15.966862 & 0.747149 & -0.137838 \\ 0.704376 & 0.747149 & 10.652406 & -3.916997 \\ -1.945778 & -0.137838 & -3.916997 & 14.161612 \end{bmatrix}.$$

Finally, it can be shown (Matlab) that the new set of eigenvalues is

$$\Omega = \{\pm 0.6700i, \pm 0.9037i, \pm 1.1275i, \pm 2.8395i\}$$

and furthermore

$$\|M_{new}Z_2\Lambda_2^2 + C_{new}Z_2\Lambda_2 + K_{new}Z_2\|_F = 6.6886 \cdot 10^{-14}$$

which shows that the embedding strategy was successful.

Numerical Example 2. Consider the vibrating system whose system matrices $M \in \mathbb{R}^{66\times 66}$ and $K \in \mathbb{R}^{66\times 66}$ come from the Harwell-Boeing BSSTRUC01 matrix set, precisely from BCSSTM02 and BCSSTK02, which concern an Statically Condensed Oil Rig Model. Besides, the damping matrix (whose measurement is not available) is set to be

$$C = \mu \begin{bmatrix} 0 & 1 & 0 & \dots & \dots & \dots \\ -1 & 0 & 1 & 0 & \dots & \dots \\ 0 & -1 & 0 & 1 & 0 & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & 0 & -1 & 0 & 1 \\ \dots & \dots & \dots & 0 & -1 & 0 \end{bmatrix}_{66\times 66}$$

where $\mu = 1.23 \times 10^0$.

This model has eigenvalues of smallest magnitude $\lambda_1 = 2.806226507i$ and $\overline{\lambda_1} = -2.806226507i$ which lie in a dangerous region, so that we want to replace them by $\sigma_1 = 9.6i$ and $\overline{\sigma_1} = -9.6i$.

Following the algorithm, steps 1 and 2 give

$$\underline{\Lambda_1} = \begin{bmatrix} 0.0000 & 2.8062 \\ -2.8062 & 0.0000 \end{bmatrix}$$
$$\Pi_1 = \begin{bmatrix} 63.974839 & -0.039463 \\ -0.039463 & 63.894879 \end{bmatrix}$$
$$D_1 = \begin{bmatrix} 7.999440 & 0. \\ 0. & 7.992414 \end{bmatrix}$$

$$Z_1 = \begin{bmatrix} -0.712678 & -0.640962 \\ 0.697330 & -0.701307 \\ 0.001697 & 0.120621 \\ -0.720456 & -0.648229 \\ 0.657928 & -0.665183 \\ -0.117963 & 0.006934 \\ -0.673474 & -0.690735 \\ 0.704843 & -0.709364 \\ 0.118349 & 0.004167 \\ -0.666154 & -0.682851 \\ 0.650700 & -0.657982 \\ -0.001221 & -0.109944 \\ -0.663761 & -0.607552 \\ 0.649824 & -0.663301 \\ 0.001686 & 0.099905 \\ -0.668279 & -0.611717 \\ 0.611432 & -0.629070 \\ -0.097642 & 0.010653 \\ -0.625771 & -0.652075 \\ 0.654052 & -0.667944 \\ 0.098388 & 0.008623 \\ -0.621624 & -0.647971 \\ 0.607390 & -0.625457 \\ -0.000834 & -0.081517 \\ -0.608429 & -0.557101 \\ 0.596319 & -0.606919 \\ 0.001671 & 0.086054 \\ -0.607471 & -0.556032 \\ 0.558396 & -0.570370 \\ -0.075429 & 0.012929 \\ -0.570690 & -0.588627 \\ 0.595130 & -0.605864 \\ 0.076504 & 0.011540 \\ -0.571720 & -0.590047 \\ 0.559419 & -0.572072 \\ -0.000465 & -0.061173 \\ -0.548375 & -0.509627 \\ 0.538044 & -0.553315 \\ 0.001617 & 0.071769 \\ -0.551016 & -0.511257 \\ 0.509209 & -0.526643 \\ -0.062326 & 0.014805 \\ -0.519874 & -0.540708 \\ 0.540441 & -0.555214 \\ 0.063685 & 0.013743 \\ -0.517445 & -0.539094 \\ 0.506775 & -0.525268 \\ -0.000182 & -0.043218 \\ -0.535866 & -0.524863 \\ 0.525493 & -0.539853 \\ 0.037257 & 0.112015 \\ -0.490174 & -0.461329 \\ 0.481269 & -0.498864 \\ 0.001593 & 0.062557 \\ -0.488772 & -0.459543 \\ 0.454691 & -0.473700 \\ -0.048130 & 0.015433 \\ -0.463565 & -0.482734 \\ 0.479989 & -0.496965 \\ 0.049676 & 0.014655 \\ -0.464627 & -0.484172 \\ 0.455865 & -0.475269 \\ 0.000095 & -0.031672 \\ -0.476527 & -0.473160 \\ 0.467787 & -0.487389 \\ 0.037566 & 0.110745 \end{bmatrix}$$

$$
Y_1 = \begin{bmatrix}
0.051998 & -0.108046 \\
-0.113931 & -0.048050 \\
0.005529 & 0.014042 \\
0.052553 & -0.109257 \\
-0.107659 & -0.045741 \\
0.013971 & -0.004801 \\
0.045102 & -0.111945 \\
-0.115182 & -0.048626 \\
-0.013489 & 0.006104 \\
0.044630 & -0.110685 \\
-0.106482 & -0.045250 \\
-0.005077 & -0.012784 \\
0.047927 & -0.101855 \\
-0.106633 & -0.045908 \\
0.004547 & 0.011644 \\
0.048252 & -0.102552 \\
-0.100568 & -0.043769 \\
0.011798 & -0.003405 \\
0.041420 & -0.105204 \\
-0.107342 & -0.046244 \\
-0.010969 & 0.005672 \\
0.041136 & -0.104532 \\
-0.099929 & -0.043543 \\
-0.003773 & -0.009475 \\
0.043922 & -0.093387 \\
-0.097769 & -0.041923 \\
0.003891 & 0.010040 \\
0.043862 & -0.093218 \\
-0.091649 & -0.039494 \\
0.009337 & -0.002087 \\
0.038062 & -0.095244 \\
-0.097582 & -0.041858 \\
-0.008300 & 0.004970 \\
0.038113 & -0.095457 \\
-0.091848 & -0.039643 \\
-0.002850 & -0.007103 \\
0.039231 & -0.085039 \\
-0.088485 & -0.038487 \\
0.003219 & 0.008384 \\
0.039459 & -0.085353 \\
-0.083885 & -0.036769 \\
0.007911 & -0.001247 \\
0.034459 & -0.087283 \\
-0.088853 & -0.038593 \\
-0.006713 & 0.004616 \\
0.034255 & -0.086981 \\
-0.083538 & -0.036726 \\
-0.002030 & -0.005011 \\
0.037061 & -0.086209 \\
-0.086395 & -0.037525 \\
0.001008 & 0.014736 \\
0.034792 & -0.076684 \\
-0.079335 & -0.034881 \\
0.002785 & 0.007317 \\
0.034715 & -0.076411 \\
-0.075067 & -0.033231 \\
0.006299 & -0.000500 \\
0.030699 & -0.077898 \\
-0.079097 & -0.034722 \\
-0.005049 & 0.004056 \\
0.030753 & -0.078115 \\
-0.075277 & -0.033357 \\
-0.001514 & -0.003662 \\
0.032652 & -0.077405 \\
-0.077231 & -0.034193 \\
0.000912 & 0.014603
\end{bmatrix}
$$

Also, in Step 2, we have

$$
\Lambda_1 = \begin{bmatrix} 0. & -2.808693 \\ 2.803762 & 0. \end{bmatrix}
$$

$$
\Theta_1 = 10^{-1} \begin{bmatrix} 0.226105 & 0. \\ 0. & 0.224269 \end{bmatrix}
$$

In Step 3 from two possibilities, we choose $\xi = 89.430565$ and compute $\eta = 87.603764$, and hence

$$
E_M = \begin{bmatrix} 89.430565 & 0. \\ 0. & 87.603764 \end{bmatrix}
$$

$$
E_C = \begin{bmatrix} 0. & -31.840631 \\ 31.245080 & 0. \end{bmatrix}
$$

$$
E_K = \begin{bmatrix} 11.143985 & 0. \\ 0. & 11.336457 \end{bmatrix}
$$

The matrices M_{new}, C_{new} and K_{new} cannot be shown here because of space limitation. However, we have used **Matlab** to compute original and updated set of eigenvalues, which are shown below.

$$
\Omega = \left\{ \begin{array}{r}
2.806227i \\ \pm 6.927072i \\ \pm 14.231601i \\ \pm 15.886123i \\ \pm 29.372872i \\ \pm 41.868654i \\ \pm 51.757022i \\ \pm 59.833982i \\ \pm 67.045518i \\ \pm 75.336585i \\ \pm 83.691327i \\ \pm 84.846413i \\ \pm 98.152377i \\ \pm 103.743200i \\ \pm 104.377662i \\ \pm 120.098563i \\ \pm 121.802030i \\ \pm 123.536003i \\ \pm 131.150184i \\ \pm 139.245604i \\ \pm 144.114161i \\ \pm 156.264174i \\ \pm 157.666560i \\ \pm 159.535589i \\ \pm 162.121383i \\ \pm 165.793160i \\ \pm 170.244514i \\ \pm 171.740139i \\ \pm 173.594641i \\ \pm 176.495116i \\ \pm 179.591564i \\ \pm 188.906549i \\ \pm 192.354778i \\ \pm 193.903979i \\ \pm 195.703015i \\ \pm 201.257113i \\ \pm 206.733465i \\ \pm 210.922015i \\ \pm 214.773292i \\ \pm 217.569590i \\ \pm 224.765421i \\ \pm 227.419053i \\ \pm 241.483031i \\ \pm 255.531930i \\ \pm 259.258276i \\ \pm 262.774938i \\ \pm 272.720454i \\ \pm 281.637264i \\ \pm 296.155378i \\ \pm 300.539577i \\ \pm 317.920243i \\ \pm 318.481185i \\ \pm 353.228533i \\ \pm 364.170234i \\ \pm 366.304868i \\ \pm 372.197723i \\ \pm 379.827544i \\ \pm 395.189960i \\ \pm 413.2257785i \\ \pm 439.221906i \\ \pm 445.583913i \\ \pm 451.677920i \\ \pm 466.231731i \\ \pm 474.375234i \\ \pm 528.655979i \\ \pm 590.156592i
\end{array} \right\}, \quad \Omega_{new} = \left\{ \begin{array}{r}
\pm 6.927072i \\ \pm 9.600000i \\ \pm 14.231601i \\ \pm 15.886123i \\ \pm 29.372872i \\ \pm 41.868654i \\ \pm 51.757022i \\ \pm 59.833982i \\ \pm 67.045518i \\ \pm 75.336585i \\ \pm 83.691327i \\ \pm 84.846413i \\ \pm 98.152377i \\ \pm 103.743200i \\ \pm 104.377662i \\ \pm 120.098563i \\ \pm 121.802030i \\ \pm 123.536003i \\ \pm 131.150184i \\ \pm 139.245604i \\ \pm 144.114161i \\ \pm 156.264174i \\ \pm 157.666560i \\ \pm 159.535589i \\ \pm 162.121383i \\ \pm 165.793160i \\ \pm 170.244514i \\ \pm 171.740139i \\ \pm 173.594641i \\ \pm 176.495116i \\ \pm 179.591564i \\ \pm 188.906549i \\ \pm 192.354778i \\ \pm 193.903979i \\ \pm 195.703015i \\ \pm 201.257113i \\ \pm 206.733465i \\ \pm 210.922015i \\ \pm 214.773292i \\ \pm 217.569590i \\ \pm 224.765421i \\ \pm 227.419053i \\ \pm 241.483031i \\ \pm 255.531930i \\ \pm 259.258276i \\ \pm 262.774938i \\ \pm 272.720454i \\ \pm 281.637264i \\ \pm 296.155378i \\ \pm 300.539577i \\ \pm 317.920243i \\ \pm 318.481185i \\ \pm 353.228533i \\ \pm 364.170234i \\ \pm 366.304868i \\ \pm 372.197723i \\ \pm 379.827544i \\ \pm 395.189960i \\ \pm 413.2257785i \\ \pm 439.221906i \\ \pm 445.583913i \\ \pm 451.677920i \\ \pm 466.231731i \\ \pm 474.375234i \\ \pm 528.655979i \\ \pm 590.156592i
\end{array} \right\}
$$

In Figures 1 and 2, the plot shows us the magnitude of the changes made to all elements of matrices M and K, respectively.

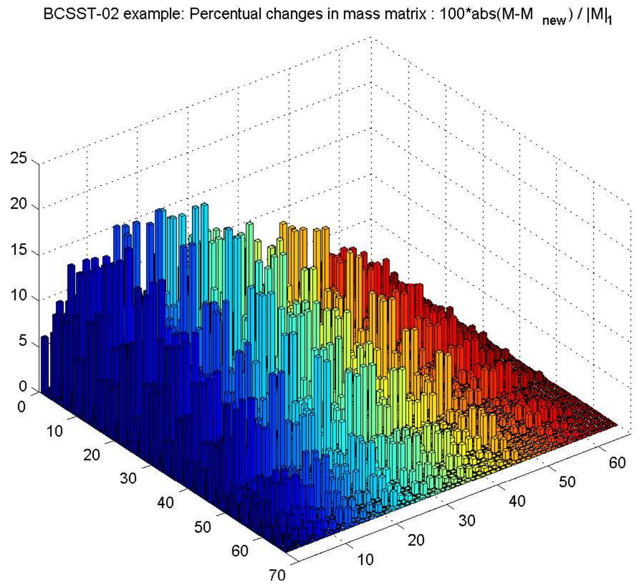

Figure 1 – Changes in mass matrix of Example 2 for the update.

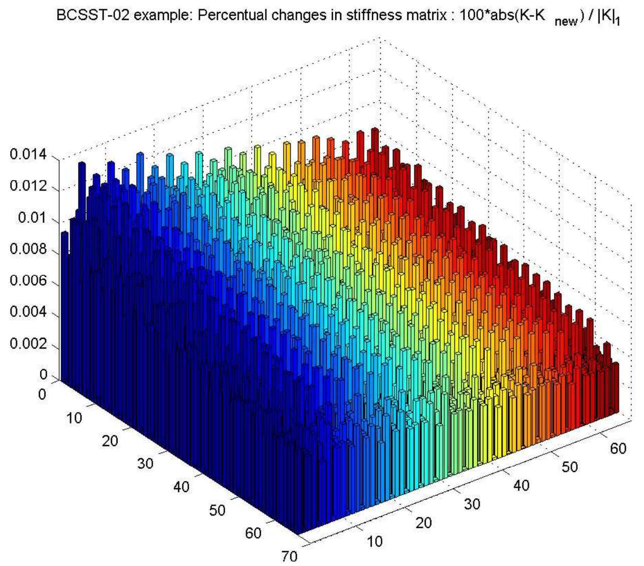

Figure 2 – Changes in stiffness matrix of Example 2 for the update.

5. CONCLUSION

The skew-symmetric eigenvalue embedding problem addressed in this paper is the one of updating a skew-symmetric finite element generated second-order model in such a way that the updated model remains skew-symmetric, and a small subset of unwanted eigenvalues is replaced by a suitably user-chosen set, while the remaining large number of eigenvalues and eigenvectors do not change. The proposed method computes new system matrices to accomplish this task. Numerical examples using real-life data from the *MatrixMarket repository*, which show the efficiency of the method, are given. The results of this paper contribute to progress in the solution of the well-known *finite-element model updating problem*.

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