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Bimeromorphic Invariants of Foliations

Abstract

Let $\mathcal{F}$ be a singular holomorphic foliation of a compact complex surface M. A bimeromorphic transformation $\phi : N \to M$ is given by a biholomorphism $\phi_{|N-\Sigma} : N - \Sigma \to M - S$, where Σ and S are analytic subsets and the basic example is the blowing-up of a point $p \in M$.

In this work are defined numerical bimeromorphic invariants of foliations. Formulas for computation and geometrical interpretations are given for the invariant $g(\mathcal{F})$, a kind of geometrical genus of the foliation.

If $\mathcal{F}$ is a foliation of the projective plane, then $g(\mathcal{F})$ is a function of the degree of the foliation and of local indices of the singularities.

In the case $\mathcal{F}$ has a rational first integral, are established inequalities between $g(\mathcal{F})$ and the geometrical genus $g(C)$ of a generic invariant compact curve.

For a generalized curve $\mathcal{F}$, $g(\mathcal{F})$ is a topological invariant.

It is also studied the behavior of the bimeromorphic invariant $\chi(\mathcal{F}) := 2\chi(\mathcal{O}_M) - 2g(\mathcal{F})$ under pullback of the foliation by generically finite maps.

Resumo

Seja $\mathcal{F}$ uma folheação holomorfa singular de uma superfície compacta complexa M. Uma transformação bimeromorfa $\phi : N \to M$ é definida como um biholomorfismo $\phi_{|N-\Sigma} : N - \Sigma \to M - S$, onde Σ e S são subconjuntos analíticos. O exemplo básico é a explosão de um ponto $p \in M$.

Neste trabalho são definidos invariantes numéricos das folheações por transformações bimeromorfas. O invariante $g(\mathcal{F})$ é o análogo para folheações do gênero geométrico de curvas, sendo descrito por fórmulas e em interpretações geométricas.

Se $\mathcal{F}$ é uma folheação do plano projetivo, então $g(\mathcal{F})$ é calculado em função do grau da folheação e de índices associados às singularidades.

No caso em que $\mathcal{F}$ admite uma integral primeira racional, são provadas desigualdades entre $g(\mathcal{F})$ e o gênero geométrico de uma curva invariante irredutível e genérica.

Se $\mathcal{F}$ é uma curva generalizada, então $g(\mathcal{F})$ é um invariante topológico.

Também se estuda o comportamento do invariante bimeromorfo $\chi(\mathcal{F}) := 2\chi(\mathcal{O}_M) - 2g(\mathcal{F})$, quando se considera o "pullback" da folheação por aplicações genericamente finitas.

Contents

1 Introduction

This work is concerned with holomorphic foliations with isolated singularities of compact complex regular surfaces M ($dim_{\mathbb{C}} M = 2$). A *bimeromorphic transformation* $\phi : N \to M$ is a biholomorphism $\phi_{|N-\Sigma} : N - \Sigma \to M - S$, where Σ and S are analytic subsets. The basic example is the blowing up of a point $p \in M$, where $S = \{p\}$ and Σ is an embedded Riemann sphere. Besides the notion of transformed foliation by a blowing up, we will consider the transformed foliation $\mathcal{G} = \phi^*(\mathcal{F})$ by any bimeromorphic transformation ϕ, that is, the holomorphic foliation with isolated singularities of N extending $(\phi_{|N-\Sigma})^*(\mathcal{F}_{|M-S})$.

Several indices are associated to the singularities of foliations [BB], [CS], [GSV] [LN1], [Su] and the finite sums of the different indices along compact curves of the surface or along all the finite singular set are basic tools for the global theory of foliations [LN1], [Br1]. Such sums of indices are not invariant under bimeromorphic transformations of the surfaces. On the other hand, the group of birational transformations $\phi : M \to M$ of a *ruled* surface M (i.e., surfaces birationally equivalent to the product $N = C \times \mathbf{C}P^1$, with C a compact Riemann surface) is much larger than the group of automorphisms (for the projective plane $\mathbf{C}P^2$ it is the *Cremona group*) and ruled surfaces appear in the study of complex differential equations as natural compactifications [LN2]. These are some of the motivations for the study of bimeromorphic global properties of the foliations.

For instance, when the surface M is the projective plane $\mathbf{C}P^2$, a foliation $\mathcal{F}$ can be given in homogeneous coordinates $(x_0 : x_1 : x_2)$ by a 1-form $\Omega = \sum_{i=0}^{2} F_i(x_0 : x_1 : x_2)dx_i$, with $F_i \in \mathbb{C}[x_0, x_1, x_2]$ homogeneous, $\gcd(F_0, F_1, F_2) = 1$, with the same degree $d = d(F_i)$ and verifying $\sum_{i=0}^{2} x_i F_i = 0$. The *degree* $d(\mathcal{F})$ is defined as the sum of order of tangencies between $\mathcal{F}$ and a non invariant projective line and $d(\mathcal{F}) = d(F_i) - 1$. A bimeromorphic transformation $\Psi : \mathbf{C}P^2 \to \mathbf{C}P^2$ is a *birational transformation* and amounts to introduce new homogeneous coordinates $(w_0 : w_1 : w_2)$ verifying, in a Zariski open subset of $\mathbf{C}P^2$,

$$\begin{cases} (w_0 : w_1 : w_2) = (Q_0(x_0 : x_1 : x_2) : Q_1(x_0 : x_1 : x_2) : Q_2(x_0 : x_1 :_2)) \\ (x_0 : x_1 : x_2) = (P_0(w_0 : w_1 : w_2) : P_1(w_0 : w_1 : w_2) : P_2(w_0 : w_1 : w_2)), \end{cases}$$

with Q_i, P_i homogeneous polynomials with the same degree D. In general the degree of $(\Psi^{-1})^*(\mathcal{F})$ depends on $d(\Psi) := D$ and for this reason $d(\mathcal{F})$ is not a birational invariant.

Given a foliation $\mathcal{F}$ in M with finite set of singularities $Sing(\mathcal{F})$, let $\{\mathcal{U}_i\}$ be an open covering of M in which $\mathcal{F}_{|\mathcal{U}_i}$ is represented by holomorphic vector fields $\{X_i\}$ with isolated zeros, verifying $X_i = f_{ij} X_j$, $f_{ij} \in \mathcal{O}^*(\mathcal{U}_i \cap \mathcal{U}_j)$, if $\mathcal{U}_i \cap \mathcal{U}_j \neq \emptyset$. Then the *tangent bundle* of $\mathcal{F}$ in M, denoted by $T_{\mathcal{F}}$, is defined (up to isomorphism) by the transition functions f_{ij}^{-1}. Its dual bundle is denoted by $T^*_{\mathcal{F}}$.

A *reduced foliation* $\widetilde{\mathcal{F}}$ associated to $\mathcal{F}$ is a foliation having only reduced singularities in the sense of [S], obtained from $\mathcal{F}$ by means of a finite sequence of blowing ups $\sigma : \widetilde{M} \to M$. With these notations, the first result is (Theorem 3.1.1)

Theorem A: *Let $\mathcal{F}$ be a singular holomorphic foliation of a compact complex surface M. Let $\widetilde{\mathcal{F}}$ in $\widetilde{M}$ be any reduced foliation associated to $\mathcal{F}$. Then the dimensions*

$$h^j(T^*_{\widetilde{\mathcal{F}}}) := dim_{\mathbb{C}} H^j(\widetilde{M}, T^*_{\widetilde{\mathcal{F}}}) \quad j = 0,1,2$$

do not depend on the particular reduced foliation $\widetilde{\mathcal{F}}$ and are bimeromorphic invariants of $\mathcal{F}$ under bimeromorphic transformations $T : M \to N$ between non-singular surfaces.

Along this work, we give geometric interpretations and formulas for the bimeromorphic invariants

$$g(\mathcal{F}) := \sum_{j=0}^{2}(-1)^j h^j(T^*_{\mathcal{F}}) \quad \text{and} \quad \chi(\mathcal{F}) := 2\chi(\mathcal{O}_M) - 2g(\mathcal{F}).$$

When $\mathcal{F}$ is given in $\mathcal{U}_i$ by $\omega_i = 0$, with ω_i a holomorphic 1-form with isolated singularities verifying $\omega_j = g_{ij}\omega_i$, if $\mathcal{U}_i \cap \mathcal{U}_j \neq \emptyset$, then the *normal bundle* $N_{\mathcal{F}}$ is defined (up to isomorphisms) by transition functions g_{ij}. For any line bundle L of M, $c_1(L)$ denotes the first Chern class. For a pair of line bundles L, L', we identify the cup product $c_1(L) \cdot c_1(L') \in H^4(M, \mathbb{Z})$ with the integer number $< c_1(L) \cdot c_1(L'), [M] >$, where $[M] \in H_4(M, \mathbb{Z})$ is the fundamental class of M. We prove (Theorem 3.2.3):

Theorem B: *If $\mathcal{F}$ is a singular holomorphic foliation of a compact complex surface M, then*

$$g(\mathcal{F}) = \chi(\mathcal{O}_M) + \frac{1}{2}c_1(T^*_{\mathcal{F}}) \cdot c_1(N_{\mathcal{F}}) - \sum_{p \in Sing(\mathcal{F})} \delta_p(\mathcal{F}),$$

where $\delta_p(\mathcal{F}) \in \mathbb{N}$ is an index (defined in Section 3.2) associated to the singularities.

In particular, for a foliation $\mathcal{F}$ in $\mathbf{C}P^2$ with degree $d(\mathcal{F})$,

$$g(\mathcal{F}) = \frac{d(\mathcal{F})(d(\mathcal{F})+1)}{2} - \sum_{p \in Sing(\mathcal{F})} \delta_p(\mathcal{F}).$$

In Section 3.4, we consider *topologically equivalent* foliations. A foliation $\mathcal{F}$ of a compact complex surface is a *generalized curve*, when each singularity of $\mathcal{F}$ is a generalized curve in the sense of [CLNS]. We prove (Theorem 3.4.1):

Theorem C: *If $\mathcal{F}$ is a generalized curve of a compact complex surface M, then $g(\mathcal{F})$ is a topological invariant.*

In Section 4 we consider foliations with *rational* first integral of projective surfaces, that is, *pencils* of curves. Among the invariant compact curves of a pencil we distinguish *generic* and *critical* (non generic) curves. Denoting $g(C)$ the *geometrical genus* of a curve C, that is, the topological genus of a normalization of C, and $\chi(C) = 2 - 2g(C)$, we prove (see Corollaries 4.2.3, 4.2.5 and 4.2.9):

Theorem D: *Let C be an irreducible generic curve of a pencil $\mathcal{F}$ of a projective surface M. Then*

i) if C is a rational curve, then $\mathcal{F}$ is birationally equivalent to the pencil of projective lines in $\mathbf{C}P^2$ containing a point and $\chi(\mathcal{F}) = 4$ (that is, $g(\mathcal{F}) = -1$);

ii) if C is an elliptic curve, then $\chi(\mathcal{F}) \geq 0$;

iii) suppose $\chi(C) \leq -2$ and suppose that each critical curve C_λ of $\mathcal{F}$ has no rational component and $(C_\lambda)_{red}$ has at most nodal points. Then

$$\chi(\mathcal{F}) \geq 2\chi(C) + K,$$

where K is the number of critical curves of $\mathcal{F}$ having some multiple component.

In Section 4.4 we relate the previous results of Section 4 to the *Poincaré problem for pencils* [P], namely, the problem of giving an upper bound to the degree $d(C)$ of an irreducible generic curve C of a pencil $\mathcal{F}$ in $\mathbf{C}P^2$ in terms of the degree $d(\mathcal{F})$. The general Poincaré problem of bounding the degree of an invariant curve by the degree of the foliation has no positive answer in general, being necessary some hypotheses either on the singularities of the invariant curves [CeLN], [CaC] or on the singularities of the foliation [Ca2].

For a critical curve $C_\lambda = \sum_i n_i C_i$, with C_i reduced and irreducible, $(C_\lambda)_{red} := \sum_i C_i$. We prove (Theorem 4.4.3)

Theorem E: *Let $\mathcal{F}$ be a pencil of $\mathbf{C}P^2$ with irreducible generic curve C. Suppose that each critical curve C_λ has no rational components and that $(C_\lambda)_{red}$ has at most nodal points. If $g(C) \geq 2$, then*

$$d(C) \leq \frac{1}{2}(d(\mathcal{F})+2) + \frac{1}{2}((d(\mathcal{F})+2)(\chi(\mathcal{F}) - 2\chi(C))(5 - 3\chi(C))).$$

In Section 5 we consider foliations with negative $g(\mathcal{F})$. We show that a foliation $\mathcal{F}$ of $\mathbf{C}P^2$ with $g(\mathcal{F}) < 0$ has a point with infinite number of local separatrices (Lemma 5.2.2) and that a generalized curve $\mathcal{F}$ in $\mathbf{C}P^2$ with $d(\mathcal{F}) = 1$ and $g(\mathcal{F}) < 0$ is a rational pencil (Proposition 5.2.1). Nevertheless, we give examples of non birationally equivalent pencils $\mathcal{F}$ in $\mathbf{C}P^2$ with $g(\mathcal{F}) = -1$ (Example 5.1.2).

Theorem E and remarks of Section 5 motivate the following birational problem, for a foliation $\mathcal{F}$ satisfying the hypotheses of Theorem V and $g(\mathcal{F}) \geq 2$:

Problem: *Are there upper bounds to the geometrical genus $g(C)$ in terms of $g(\mathcal{F})$?*

Lemma 5.2.2 motivates the next question.

Question: *Are there always rational first integrals for foliations of $\mathbf{C}P^2$ with negative $g(\mathcal{F})$?*

We state a conjecture:

Conjecture 1: *Let $\mathcal{F}$ be a generalized curve in $\mathbf{C}P^2$ with $g(\mathcal{F}) < 0$. Then $g(\mathcal{F}) = -1$.*

We show in Section 5 that Conjecture 2 implies Conjecture 1.

Conjecture 2: *Let $\mathcal{G}$ be a connected rational fibration over $\mathbf{C}P^1$, given by $g : M \to \mathbf{C}P^1$. Suppose that the singular fibers of $\mathcal{G}$ have as sets at most nodal singularities. Then for any reduced foliation $\mathcal{F}$ of M, then $Det(\mathcal{F}) \geq Det(\mathcal{G})$, where $Det()$ is the sum of Milnor numbers.*

Conjecture 2 is motivated by the following result:

Theorem: [Br1] *Let $\mathcal{G}$ be a rational or elliptic connected fibration over a compact Riemann surface C, given by $g : M \to C$. If $Det(\mathcal{G}) > 0$, then $Det(\mathcal{F}) > 0$ for any foliation $\mathcal{F}$ of M.*

In Section 6 it is studied the behavior of $\chi(\mathcal{F})$ under pullback by *generically finite* maps. We deal with foliations of singular surfaces and apply the *canonical desingularization* to the surface. The result obtained (Theorem 6.4.3) generalizes the following:

Theorem F: *Let $\pi : N \to M$ be a 2-fold covering of a compact regular surface M, ramified along a curve $C \subset M$, C with generalized cusps singularities. Let $\phi : N' \to N$ be a canonical desingularization of the surface N and $p = \pi \circ \phi : N' \to M$.*

If C is in general position with respect to a foliation $\mathcal{F}$ of M, then

$$\chi(p^*(\mathcal{F})) = 2\chi(\mathcal{F}) - c_1(N_{\mathcal{F}}) \cdot c_1(\mathcal{O}(C)).$$

In particular, for $M \cong \mathbf{C}P^2$ and a curve C of even degree, $\chi(p^*(\mathcal{F})) = 2\chi(\mathcal{F}) - (d(\mathcal{F})+2)d(C)$.

2 Preliminaries

We fix notations and recall some properties of line bundles and intersection numbers.

For a complex vector bundle L on a compact complex surface M, $c_i(L) \in H^{2i}(M, \mathbb{Z})$ denotes the i-th Chern class. For L, L' complex line bundles of a compact complex surface M, and V a complex vector bundle of M, we identify the cup product $c_1(L) \cdot c_1(L') \in H^4(M, \mathbb{Z})$ and $c_2(V) \in H^4(M, \mathbb{Z})$ with the integer numbers $< c_1(L) \cdot c_1(L'), [M] >$, $< c_2(V), [M] >$, where $[M] \in H_4(M, \mathbb{Z})$ is the fundamental class of M. With such identification, the *intersection number* $D \cdot D'$ of a pair of divisors D and D', is defined by

$$D \cdot D' := c_1(\mathcal{O}(D)) \cdot c_1(\mathcal{O}(D')),$$

where $\mathcal{O}(D)$ denotes the line bundle of M associated to a divisor $D \subset M$. We consider also the product $c_1(L) \cdot C := c_1(L) \cdot c_1(\mathcal{O}(C))$.

Let σ be a blowing up at $p \in M$. For the exceptional line $E = \sigma^{-1}(p)$, $E \cdot E = -1$. For line bundles L and L' of M,

$$c_1(\sigma^*(L)) \cdot c_1(\mathcal{O}(E)) = 0,$$

and

$$c_1(\sigma^*(L)) \cdot c_1(\sigma^*(L')) = c_1(L) \cdot c_1(L').$$

2.1 Bimeromorphic invariants of curves and surfaces

The references for all the preliminaries about bimeromorphic geometry of curves and surfaces are [Be] and [BPV].

If N is a n-dimensional compact complex manifold, $K_N = \bigwedge^n T_N^*$ denotes the canonical line bundle of N, that is, the line bundle whose local holomorphic sections are given by holomorphic n-forms. We also consider the cohomology groups $H^j(M, L)$, where we identify L with its sheaf of sections and $h^j(,) := dim_{\mathbb{C}} H^j(,)$.

The Structure Theorem for bimeromorphic transformations between regular surfaces asserts that, for any bimeromorphic transformation $T : N \to M$, there are a surface Z and finite sequences of *blowing ups* $\sigma_1 : Z \to N$ and $\sigma_2 : Z \to M$, such that $T = \sigma_2 \circ \sigma_1^{-1}$.

Let $\nu_p(C) \geq 0$ denote the *algebraic multiplicity* of an analytic curve at a point p, that is the order of the first non-zero jet in its Taylor series. The *strict transform* $\widetilde{C}$ of C by a blowing up σ at p is defined by $\widetilde{C} := \sigma^*(C) - \nu_p(C)E$, with $E = \sigma^{-1}(p)$. The Structure Theorem enables us to define the strict transform of a curve by any bimeromorphic transformation.

When the surfaces are projective, by Chow's Theorem the bimeromorphic transformations are *birational* transformations. The group of birational transformations in $\mathbb{C}P^2$ is the Cremona group.

Given a n-dimensional compact complex connected manifold N, its *geometrical genus* $p_g(N)$ is defined by $p_g(N) := h^n(N, \mathcal{O}_N)$. By Serre's Duality, $p_g(N) = h^0(N, K_N)$, that is, $p_g(N)$ is the number of linearly independent holomorphic n-forms. For a compact Riemann surface C, $p_g(C)$ is equal to the *topological genus* of C and will be denoted in this work by $g(C)$. For a compact Riemann surface C embedded in a surface N, it follows from Riemann-Roch's Theorem that $g(C) = 1 + \frac{1}{2}(C \cdot C + C \cdot K_N)$.

For any compact curve $C \subset N$, the *arithmetical genus*, $p_a(C)$, is defined by $p_a(C) := 1 + \frac{1}{2}(C \cdot C + C \cdot K_N)$. If $f : \widetilde{C} \to C$ is a *desingularization* of a compact curve C, the *geometrical genus* of C is $g(C) := g(\widetilde{C})$.

Let p be a singular point of an analytic curve C. If $\pi : \widetilde{C} \to C$ is a *normalization* of (C, p) with $\pi^{-1}(p) = \{p_1, ..., p_r\}$, by composition with π there is an injective morphism $\pi^* : \mathcal{O}_{C,p} \to \mathcal{O}_{\widetilde{C},p_i}$ and it is defined $\delta_p(C) := dim_{\mathbb{C}}(\oplus_{i=1}^r \mathcal{O}_{\widetilde{C},p_i}/\pi^*(\mathcal{O}_{C,p}))$. For C compact, it holds

$$g(C) = p_a(C) - \sum_{p \in Sing(C)} \delta_p.$$

Let $f_i : N_i \to N_{i-1} \to \cdots \to N_1 \to N$ be a finite sequence of blowing ups, where $\sigma_1 : N_1 \to N$ is a blowing up at $p \in C$ and, for $i \geq 2$, $\sigma_i : N_i \to N_{i-1}$ denotes the blowing up of possibly several points. Let $\widetilde{C}^{(i)} \subset N_i$ be the strict transform of an analytic curve $C \subset N$ by f_i and $D_i := f_i^{-1}(p)$. Suppose that $\forall j \geq 2$ and $i \geq j$, $\widetilde{C}^{(j)}$ results from $\widetilde{C}^{(j-1)}$ by blowing ups of the intersection points $D_{j-1} \cap \widetilde{C}^{(j-1)}$. Then the points of $D_i \cap \widetilde{C}^{(i)}$ are the *infinitely near points* of p relatively to f_i. If $\widetilde{C} := \widetilde{C}^{(n)}$ is smooth for some n, define $Sing\mathcal{R}(C, p)$ as the set composed by p and all infinitely near points relatively to f_i for $i = 1, \cdots, n$. Denoting $\nu_q := \nu_q(\widetilde{C}^{(i)})$, it is proved that

$$\delta_p(C) = \sum_{q \in Sing\mathcal{R}(C,p)} \frac{\nu_q(\nu_q - 1)}{2}.$$

In the case of a compact complex surface N ($dim_{\mathbb{C}} N = 2$), the irregularity of N, denoted $q(N)$, is defined by $q(N) := h^1(N, \mathcal{O}_N)$. The i-th plurigenus of a surface N is given by:

$$p_i(N) := h^0(N, K_N^{\otimes i}),$$

that is, in particular, $p_1(N) = p_g(N)$. The numbers $p_i(N)$, $q(N)$ are invariants of bimeromorphic transformations between smooth compact surfaces.

Recall M. Noether's Formula for $\chi(\mathcal{O}_N) := \sum_{i=0}^{2}(-1)^i h^i(N, \mathcal{O}_N)$:

$$\chi(\mathcal{O}_N) = \frac{1}{12}(c_1^2(N) + c_2(N)).$$

2.2 Line bundles associated to foliations

For the facts stated without proof in this section, we refer to [Cam], [CaC] and [Br1].

Given a foliation $\mathcal{F}$ of a compact complex surface M, let $\{\mathcal{U}_i\}$ be an open covering of M in which $\mathcal{F}_{|\mathcal{U}_i}$ is represented by local vector fields $\{X_i\}$ with $X_i = f_{ij}X_j$, $f_{ij} \in \mathcal{O}^*(\mathcal{U}_i \cap \mathcal{U}_j)$, for each $\mathcal{U}_i \cap \mathcal{U}_j \neq \emptyset$. Then, the *tangent bundle* of $\mathcal{F}$ in M, denoted $T_{\mathcal{F}}$, is defined (up to isomorphism) by the transition functions f_{ij}^{-1}. The foliation $\mathcal{F}$ is determined by a bundle map $f : T_{\mathcal{F}} \to TM$ such that:

i) $f((T_{\mathcal{F}})_p) \subset T_pM$ and

ii) f is injective if and only if $p \notin Sing(\mathcal{F})$ (in this case $f((T_{\mathcal{F}})_p)$ is the complex tangent line to $\mathcal{F}$ at p).

A (meromorphic) holomorphic section s of $T_{\mathcal{F}}$ gives by composition $X = f \circ s$ a (meromorphic) holomorphic vector field in M generating $\mathcal{F}$.

If M is a projective (algebraic) compact complex surface and X is a meromorphic vector field in M generating $\mathcal{F}$, then the divisor associated to $T_{\mathcal{F}}$ is $(X)_0 - (X)_\infty$.

The *normal bundle* of $\mathcal{F}$ in M, denoted $N_{\mathcal{F}}$, is defined by means of 1-forms instead of vector fields: it is defined (up to isomorphism) by the transition functions $g_{ij} \in \mathcal{O}^*(U_i \cap U_j)$ given by $\omega_i = g_{ij}\omega_j$, where $\omega_i = 0$ represents $\mathcal{F}$ in $\mathcal{U}_i$. If $N^*_{\mathcal{F}}$ denotes the dual bundle of $N_{\mathcal{F}}$, then the foliation is determined by a bundle map $g : N^*_{\mathcal{F}} \to TM^*$. A (meromorphic) holomorphic section s of $N^*_{\mathcal{F}}$ gives by composition a (meromorphic) holomorphic 1-form $\Omega = g \circ s$ generating $\mathcal{F}$.

The relation between these line bundles is given by:

$$K_M = T^*_{\mathcal{F}} \otimes N^*_{\mathcal{F}}. \tag{1}$$

Let $\sigma : \widetilde{M} \to M$ denote a blowing up at $p \in M$ and $E = \sigma^{-1}(p) \subset \widetilde{M}$ the exceptional line. The foliation $\sigma^*(\mathcal{F}|_{M-\{p\}})$ of $\widetilde{M} - E$ has an unique extension to a foliation $\widetilde{\mathcal{F}}$ with isolated singularities of $\widetilde{M}$, the *strict transform* of $\mathcal{F}$ by σ. By the Structure Theorem it is possible to define the strict transform of a foliation by any bimeromorphic transformation.

If $\mathcal{F}$ is represented locally by $\omega = 0$, where ω is a holomorphic 1-form with isolated zero at p, then $\widetilde{\omega}=\sigma^*(\omega)$ has along the exceptional line E a line of zeros of order $m_p \geq 0$. The order of the first non-zero jet of ω at p is the *algebraic multiplicity* of $\mathcal{F}$ at p, denoted $\nu(\mathcal{F},p)$, and

$$m_p = \begin{cases} \nu_p(\mathcal{F}) & \text{if p is non-dicritical} \\ \nu_p(\mathcal{F})+1 & \text{if p is dicritical,} \end{cases}$$

where *dicritical* means, in all this work, a singularity p of $\mathcal{F}$ such that the exceptional line E of a blowing up σ is not invariant by $\widetilde{\mathcal{F}}$ (we call the attention to the fact that in the literature often a singularity is called dicritical when some component of the exceptional divisor of its resolution is not invariant by the transformed foliation).

The effect of a blowing up σ on the line bundles $T^*_{\mathcal{F}}$ and $N_{\mathcal{F}}$ is:

$$T^*_{\widetilde{\mathcal{F}}} = \sigma^*(T^*_{\mathcal{F}}) \otimes \mathcal{O}((1-m_p)E) \quad \text{and} \quad N_{\widetilde{\mathcal{F}}} = \sigma^*(N_{\mathcal{F}}) \otimes \mathcal{O}(-m_pE), \tag{2}$$

which implies

$$\begin{aligned} c_1^2(N_{\widetilde{\mathcal{F}}}) &= c_1^2(\sigma^*(N_{\mathcal{F}})) + c_1^2(\mathcal{O}(-m_pE)) \\ &= c_1^2(N_{\mathcal{F}}) - m_p^2, \end{aligned}$$

2.3 Singularities of foliations

If $p \in M$ is an isolated singular point of $\mathcal{F}$, let (x,y) be local coordinates with $p=(0,0)$ such that the foliation is represented by

$$X(x,y) = P(x,y)\frac{\partial}{\partial x} + Q(x,y)\frac{\partial}{\partial y}, \quad \gcd(P,Q) = 1$$

with $J(x,y)$ the Jacobian matrix of (P,Q). Define

$$Det(\mathcal{F},p) := Res_0\{\frac{det J(x,y)}{P(x,y)Q(x,y)} dx \wedge dy\}$$

and

$$Tr(\mathcal{F},p) := Res_0\{\frac{(tr J(x,y))^2}{P(x,y)Q(x,y)} dx \wedge dy\},$$

where $Res_0\{\}$ means the residue at $(0,0)$ of the meromorphic 2-form. Remark that $Det(\mathcal{F},p)$ is the *Milnor number* of the singular point p of $\mathcal{F}$:

$$Det(\mathcal{F},p) = dim_{\mathbb{C}} \frac{\mathcal{O}_{(\mathbb{C}^2,0)}}{< P(x,y), Q(x,y) >}.$$

According to Baum-Bott's Theorem [BB]:

$$Det(\mathcal{F}) := \sum_{p\in Sing(\mathcal{F})} Det(\mathcal{F},p) = c_2(M) + c_1^2(T^*_{\mathcal{F}}) + c_1(T^*_{\mathcal{F}}) \cdot c_1(M) \tag{3}$$

$$Tr(\mathcal{F}) := \sum_{p\in Sing(\mathcal{F})} Tr(\mathcal{F},p) = c_1^2(M) + c_1^2(T^*_{\mathcal{F}}) + 2c_1(T^*_{\mathcal{F}}) \cdot c_1(M), \tag{4}$$

which implies

$$Tr(\mathcal{F}) = c_1^2(N_{\mathcal{F}}). \tag{5}$$

By a blowing up σ of a point p, with $E = \sigma^{-1}(p)$, we obtain:

$$Tr(\widetilde{\mathcal{F}}) = Tr(\mathcal{F}) - m_p^2$$

and [CeM], [CLNS]

$$\sum_{q\in E} Det(\widetilde{\mathcal{F}},q) = Det(\mathcal{F},p) - m_p(m_p-1)+1. \tag{6}$$

A foliation $\mathcal{F}$ is called *reduced* when each of its singularities admit local coordinates (x,y) such that $\mathcal{F}$ is represented by a holomorphic 1-form ω with 1-jet given by $\omega_1 = \mu x dy - \lambda y dx$, satisfying one of the following conditions: i) $\mu \neq 0$ and $\lambda = 0$ or vice-versa, or ii) $\mu\lambda \neq 0$ and $\frac{\lambda}{\mu} \in \mathbb{C} - \mathbb{Q}^+$. In case i) the singularity is called a *saddle-node.*

By [S], for any foliation $\mathcal{F}$ in M, there exists a finite number of blowing ups, denoted $f : \widetilde{M} \to M$, such that $\widetilde{\mathcal{F}}$, the strict transform of $\mathcal{F}$ by f, is reduced.

Let C be an analytic curve such that all its local branches at p are invariant by a local foliation $\mathcal{F}$. The index $Z(C,\mathcal{F},p)$ defined in [GSV] generalizes for singular curves the *Poincaré-Hopf index.* If C is a compact curve invariant by a foliation $\mathcal{F}$ of M, $Z(C,\mathcal{F}) := \sum_{p\in C} Z(C,\mathcal{F},p)$ and by [Br1]:

$$c_1(T^*_{\mathcal{F}})\cdot C = Z(C,\mathcal{F}) + C\cdot C + C.K_M = Z(C,\mathcal{F}) + 2p_a(C) - 2. \tag{7}$$

2.4 Order of tangency

Let C be an analytic curve and suppose that all its local branches at p are *not* invariant by a local foliation $\mathcal{F}$. The *order of tangency* at p between C and $\mathcal{F}$ is defined in [Br1] by:

$$tang(C,\mathcal{F},p) := dim_{\mathbb{C}} \frac{\mathcal{O}_{(\mathbb{C}^2,p)}}{< f, X(f) >},$$

where $f=0$ is a reduced local equation of C and X is a local holomorphic vector field with isolated zeros generating $\mathcal{F}$. If C be a compact curve of M, with all irreducible component not invariant by $\mathcal{F}$ of M, then

$$\begin{aligned} tang(C,\mathcal{F}) := \sum_p tang(C,\mathcal{F},p) &= c_1(T^*_{\mathcal{F}})\cdot C + C\cdot C && (8)\\ &= c_1(N_{\mathcal{F}})\cdot C + C\cdot C + K_M\cdot C && (9)\\ &= c_1(N_{\mathcal{F}})\cdot C + 2p_a(C) - 2. && (10)\end{aligned}$$

We call the attention to the fact that this definition of order of tangency does not coincide in the case of singular curves with the definition of [Cam] (the definitions are compared in [Se]). According to [Cam], if $\omega = 0$ represents at p the foliation $\mathcal{F}$, taking $\pi : \tilde{C} \to C$ a normalization of C with $\pi^{-1}(p) = \{q_1,...,q_n\}$, $tang'(C,\mathcal{F},p) := \sum_{j=1}^{n} ord_{q_j}\pi^*(\omega)$ and formulas analogous to formula (10) above are proved for $tang'(C,\mathcal{F}) := \sum_p tang'(C,\mathcal{F},p)$ (instead of the arithmetical genus $p_a(C)$ what appears is the geometrical genus $g(C)$). In what follows we will use the definition of [Br1].

If E is the exceptional line of a blowing up at a dicritical point p of $\mathcal{F}$, we obtain, for the strict transform foliation $\widetilde{\mathcal{F}}$:

$$m_p = tang(E,\widetilde{\mathcal{F}}) + 2. \tag{11}$$

In fact,

$$\begin{aligned} tang(E,\widetilde{\mathcal{F}}) &= c_1(N_{\widetilde{\mathcal{F}}})\cdot E + 2p_a(E) - 2\\ &= (c_1(\sigma^*(N_{\mathcal{F}})) - m_p c_1(\mathcal{O}(E)))\cdot E - 2\\ &= m_p - 2, \end{aligned}$$

Let $C \subset M$ be a non $\mathcal{F}$-invariant curve with algebraic multiplicity $\nu_p = \nu_p(C)$ at p. Let $\widetilde{C}$ and $\widetilde{\mathcal{F}}$ be the strict transforms by a blowing up σ at p of C and $\mathcal{F}$ respectively, with $E = \sigma^{-1}(p)$. Then

$$tang(\widetilde{C}, \widetilde{\mathcal{F}}) = tang(C, \mathcal{F}) - \nu_p(m_p + \nu_p - 1). \tag{12}$$

In fact,

$$\begin{aligned} tang(\widetilde{C}, \widetilde{\mathcal{F}}) &= c_1(N_{\widetilde{\mathcal{F}}}) \cdot \widetilde{C} + \widetilde{C} \cdot \widetilde{C} + \widetilde{C} \cdot K_{\widetilde{M}} \\ &= c_1(N_{\mathcal{F}}) \cdot C + C \cdot C + C \cdot K_M - \nu_p(m_p + \nu_p - 1) \\ &= tang(C, \mathcal{F}) - \nu_p(m_p + \nu_p - 1) \end{aligned}$$

3 Invariants of foliations

3.1 Definition and bimeromorphic invariance

Theorem 3.1.1 *Let $\mathcal{F}$ be a foliation of a non-singular compact complex surface M. Let $\widetilde{\mathcal{F}}$ in $\widetilde{M}$ be any reduced foliation associated to $\mathcal{F}$. Then the dimensions*

$$h^j(T^*_{\widetilde{\mathcal{F}}}) := dim_{\mathbb{C}} H^j(\widetilde{M}, T^*_{\widetilde{\mathcal{F}}}) \quad j = 0,1,2$$

do not depend on the particular reduced foliation $\widetilde{\mathcal{F}}$ and are bimeromorphic invariants of $\mathcal{F}$ under bimeromorphic transformations $T : M \to N$ between non-singular surfaces.

Supposing proved the theorem, the following bimeromorphic invariants are defined:

Definition 3.1.2 $g(\mathcal{F}) := \sum_{j=0}^{2}(-1)^j h^j(T^*_{\widetilde{\mathcal{F}}})$ and $\chi(\mathcal{F}) := 2\chi(\mathcal{O}_M) - 2g(\mathcal{F})$.

We state two consequences.

We recall to what follows that the submanifols of a Kähler manifold are also Kähler. Since the projective space $\mathbf{C}P^N$ with the Fubini-Study metric is a Kähler manifold, then every projective (algebraic) compact complex surface is Kähler. Examples of Kähler surfaces M with $q(M) := h^1(M, \mathcal{O}_M) = 0$ are the *rational* surfaces, i.e., surfaces birationally equivalent to $\mathbf{C}P^2$.

Corollary 3.1.3 *Let M be a compact Kähler surface with $q(M) := h^1(M, \mathcal{O}_M) = 0$. Let $\mathcal{F}$ be a foliation of M and let $\widetilde{\mathcal{F}}$ in $\widetilde{M}$ be a reduced associated foliation. Then*

$$g(\mathcal{F}) = h^0(T^*_{\widetilde{\mathcal{F}}}) - h^1(T^*_{\widetilde{\mathcal{F}}}).$$

In particular, if $g(\mathcal{F}) < 0$ then $h^1(T^*_{\widetilde{\mathcal{F}}}) > 0$.

Proof First remark that, for any foliation $\mathcal{F}$ of any compact complex surface M, $h^2(T^*_{\widetilde{\mathcal{F}}}) = h^0(N^*_{\widetilde{\mathcal{F}}})$. In fact, for any foliation $\mathcal{G}$, $T^*_{\mathcal{G}} = K_M \otimes N_{\mathcal{G}}$ and therefore, by Serre's Duality, $h^j(T^*_{\mathcal{G}}) = h^{2-j}(N^*_{\mathcal{G}})$.

If $h^2(T^*_{\widetilde{\mathcal{F}}}) = h^0(N^*_{\widetilde{\mathcal{F}}}) > 0$, then the existence of a bundle map $g : N^*_{\widetilde{\mathcal{F}}} \to (T\widetilde{M})^*$ associated to $\widetilde{\mathcal{F}}$ would imply $h^0(\widetilde{M}, \Omega^1_{\widetilde{M}}) > 0$. But since $h^1(M, \mathcal{O}_M)$ is a bimeromorphic invariant, we have by Hodge Theorem $h^0(\widetilde{M}, \Omega^1_{\widetilde{M}}) = h^1(\widetilde{M}, \mathcal{O}_{\widetilde{M}}) = 0$, a contradiction. □

Let L be a holomorphic line bundle of a compact complex surface M. If for each $x \in M$ there is at least one section $s \in \Gamma(M, L)$ with $s(x) \neq 0$, then there is a holomorphic map $\phi : M \to \mathbf{C}P^N$ determined by $\Gamma(M, L)$ (after choosing a basis), with $N = dim_{\mathbb{C}}\Gamma(M, L) - 1$. L is *very ample* if ϕ is an isomorphism onto its image. L is *ample* if there is a $k \geq 1$ such that $L^{\otimes k}$ is very ample.

Corollary 3.1.4 *Let $\mathcal{F}$ be a foliation of a compact projective surface M and let $\widetilde{\mathcal{F}}$ in $\widetilde{M}$ be any reduced associated foliation. If $N_{\widetilde{\mathcal{F}}}$ is ample in $\widetilde{M}$, then*

$$g(\mathcal{F}) = h^0(T^*_{\widetilde{\mathcal{F}}}).$$

Proof In fact, as remarked above $h^j(T^*_{\widetilde{\mathcal{F}}}) = h^{2-j}(N^*_{\widetilde{\mathcal{F}}})$ and, by Kodaira Vanishing Theorem [BPV], $h^0(N^*_{\widetilde{\mathcal{F}}}) = h^1(N^*_{\widetilde{\mathcal{F}}}) = 0$. □

The proof of Theorem 3.1.1 is based on a property of blowing ups, which we state as a lemma:

Lemma 3.1.5 *Let $\sigma : N \to M$ be a blowing up of a point $p \in M$ and L a line bundle of M. Then*

$$H^j(N, \sigma^*(L)) = H^j(M, L).$$

Proof According to [BPV] (Theorem I.9.1.ii), for the higher order direct image sheaves $R^q\sigma_*(\mathcal{O}_N)$, if $q > 0$, then $R^q\sigma_*(\mathcal{O}_N) = 0$.

By the projection formula [Ha] (Ex. II.5.1.d and Ex. III.8.3), $R^q\sigma_*(\sigma^*(L)) = 0$, if $q > 0$.

Then $H^j(N, \sigma^*(L)) = H^j(M, L)$, for $j \geq 0$, by [G] (Chap. F, Cor.6). □

Proof (Theorem 3.1.1) First we will show that $h^j(T^*_{\widetilde{\mathcal{F}}})$ do not depend on the particular reduced associated foliation $\widetilde{\mathcal{F}}$. For this, consider two reduced associated foliations $(\widetilde{M}_i, \widetilde{\mathcal{F}}_i)$, $i = 1, 2$, associated to $(M, \mathcal{F})$. Since they are bimeromorphically equivalent foliations, by the Structure Theorem there are a surface $\widetilde{M}_3$ and sequences of blowing ups, denoted $\sigma_i : \widetilde{M}_3 \to \widetilde{M}_i$, such that $\sigma_1^*(\widetilde{\mathcal{F}}_1) = \sigma_2^*(\widetilde{\mathcal{F}}_2)$.

It is enough to consider the case when each σ_i is a blowing up of p_i, with $E_i := \sigma_i^{-1}(p_i)$ the exceptional line and $\widetilde{\mathcal{F}}_3 := \sigma_i^*(\widetilde{\mathcal{F}}_i)$. By §2.2

$$T^*_{\widetilde{\mathcal{F}}_3} = \sigma_i^*(T^*_{\widetilde{\mathcal{F}}_i}) \otimes \mathcal{O}((1 - m_{p_i})E_i) \quad \text{and} \quad N^*_{\widetilde{\mathcal{F}}_3} = \sigma_i^*(N^*_{\widetilde{\mathcal{F}}_i}) \otimes \mathcal{O}(m_{p_i}E_i).$$

Since $\widetilde{\mathcal{F}}_i$ have only reduced singularities, then $m_{p_i} = 0, 1$.

If $m_{p_i} = 1$, then $T^*_{\widetilde{\mathcal{F}}_3} = \sigma_i^*(T^*_{\widetilde{\mathcal{F}}_i})$ and Lemma 3.1.5 implies that $h^j(T^*_{\widetilde{\mathcal{F}}_3}) = h^j(T^*_{\widetilde{\mathcal{F}}_i})$.

If $m_{p_i} = 0$, then $N^*_{\widetilde{\mathcal{F}}_3} = \sigma_i^*(N^*_{\widetilde{\mathcal{F}}_i})$ and, by Lemma 3.1.5,

$$h^j(N^*_{\widetilde{\mathcal{F}}_3}) = h^j(N^*_{\widetilde{\mathcal{F}}_i}) \quad j \geq 0.$$

Since, as already remarked, $h^j(T^*_{\widetilde{\mathcal{F}}_3}) = h^{2-j}(N^*_{\widetilde{\mathcal{F}}_3})$ and $h^{2-j}(N^*_{\widetilde{\mathcal{F}}_i}) = h^j(T^*_{\widetilde{\mathcal{F}}_i})$, we conclude

$$h^j(T^*_{\widetilde{\mathcal{F}}_3}) = h^j(T^*_{\widetilde{\mathcal{F}}_i}), \quad j \geq 0, \quad i = 1, 2.$$

This proves that the dimensions $h^j(T^*_{\widetilde{\mathcal{F}}})$ are well defined.

Consider now $T : N \to M$ a bimeromorphic transformation and the foliation $\mathcal{G} := (T)^*(\mathcal{F})$ of N. Let $\widetilde{\mathcal{F}}$ of $\widetilde{M}$ and $\widetilde{\mathcal{G}}$ of $\widetilde{N}$ be reduced foliations associated to $\mathcal{F}$ and $\mathcal{G}$. Since $\widetilde{\mathcal{F}}$ and $\widetilde{\mathcal{G}}$ are bimeromorphically equivalent, there exist a surface Z and sequences of blowing ups $\sigma_1 : Z \to \widetilde{M}$ and $\sigma_2 : Z \to \widetilde{N}$ such that $\sigma_1^*(\widetilde{\mathcal{F}})$ and $\sigma_2^*(\widetilde{\mathcal{G}})$ coincide. Since $\widetilde{\mathcal{F}}$ and $\widetilde{\mathcal{G}}$ are reduced foliations, the previous reasoning applies to $\widetilde{\mathcal{F}}$ and $\widetilde{\mathcal{G}}$ and suffices to prove that $h^j(T^*_{\widetilde{\mathcal{F}}}) = h^j(T^*_{\widetilde{\mathcal{G}}})$.

□

3.2 Formulas for $g(\mathcal{F})$

Let $f_i : M_i \to M_{i-1} \to \cdots \to M_1 \to M$ be a finite sequence of blowing ups, where $\sigma_1 : M_1 \to M$ is a blowing up at $p \in Sing(\mathcal{F})$ and, for $i \geq 2$, $\sigma_i : M_i \to M_{i-1}$ denotes the blowing up of possibly several points. Let $\widetilde{\mathcal{F}}^{(i)}$ be the strict transform of the foliation $\mathcal{F}$ of M by f_i and $D_i := f_i^{-1}(p)$. Suppose that $\forall j \geq 2$ and $i \geq j$, $\widetilde{\mathcal{F}}^{(j)}$ results from $\widetilde{\mathcal{F}}^{(j-1)}$ by blowing ups of the singularities of $D_{j-1} \cap \widetilde{\mathcal{F}}^{(j-1)}$. Then the singularities of $D_i \cap \widetilde{\mathcal{F}}^{(i)}$ are the *infinitely near singularities* of p relatively to f_i. With this notation, we define:

Definition 3.2.1 Let $\widetilde{\mathcal{F}} := \widetilde{\mathcal{F}}^{(n)}$ be a reduced foliation, for some n, and denote by $Sing\mathcal{R}(\mathcal{F}, p)$ the set composed by p and all infinitely near singularities relatively to f_i, for $i = 1, \cdots, n$. Then, with $m_q := m_q(\widetilde{\mathcal{F}}^{(i)})$,

$$\delta_p(\mathcal{F}) := \sum_{q \in Sing\mathcal{R}(\mathcal{F},p)} \frac{m_q(m_q - 1)}{2}.$$

Denote by $Sing\mathcal{R}(\mathcal{F})$ the union of $Sing\mathcal{R}(\mathcal{F}, p)$ for all $p \in Sing(\mathcal{F})$.

Definition 3.2.2 For a foliation $\mathcal{F}$ of a compact complex surface M,

$$p_a(\mathcal{F}) := \chi(\mathcal{O}_M) + \frac{1}{2} c_1(T^*_{\mathcal{F}}) \cdot c_1(N_{\mathcal{F}}).$$

Theorem 3.2.3 *If $\widetilde{\mathcal{F}}$ in $\widetilde{M}$ is any reduced foliation associated to $\mathcal{F}$, then*

$$i) \quad g(\mathcal{F}) = p_a(\mathcal{F}) - \sum_{p \in Sing(\mathcal{F})} \delta_p(\mathcal{F})$$

and

$$ii) \quad g(\mathcal{F}) = \chi(\mathcal{O}_M) + \frac{1}{2}(Det(\mathcal{F}) - c_2(M)) - \sum_{p \in Sing(\mathcal{F})} \delta_p(\mathcal{F}).$$

Proof By Riemann-Roch's Theorem and §2.2

$$\begin{aligned} g(\mathcal{F}) &:= \sum_{j=0}^{2} (-1)^j h^j(T^*_{\widetilde{\mathcal{F}}}) \\ &= \chi(\mathcal{O}_{\widetilde{M}}) + \frac{1}{2} c_1(T^*_{\widetilde{\mathcal{F}}}) \cdot (c_1(T^*_{\widetilde{\mathcal{F}}}) + c_1(\widetilde{M})) \\ &= \chi(\mathcal{O}_M) + \frac{1}{2} c_1(T^*_{\widetilde{\mathcal{F}}}) \cdot c_1(N_{\widetilde{\mathcal{F}}}). \end{aligned}$$

By Baum-Bott Theorem §2.3 (3),

$$g(\mathcal{F}) = \chi(\mathcal{O}_M) + \frac{1}{2}(Det(\widetilde{\mathcal{F}}) - c_2(\widetilde{M})).$$

If $\mathcal{F}'$ denotes the transformed foliation by a blowing up of $\mathcal{F}$ at p, by §2.2

$$c_1(T^*_{\mathcal{F}'}) \cdot c_1(N_{\mathcal{F}'}) = c_1(T^*_{\mathcal{F}}) \cdot c_1(N_{\mathcal{F}}) - m_p(m_p - 1),$$

and the result follows by considering the definition of $\delta_p(\mathcal{F})$ and the sequence of blowing ups of the resolution $\mathcal{R}(\mathcal{F})$.

□

The *degree* of a foliation $\mathcal{F}$ of $\mathbf{C}P^2$, denoted $d(\mathcal{F})$, is defined by

$$d(\mathcal{F}) := \sum_{p \in L} tang(L, \mathcal{F}, p),$$

where L is a non $\mathcal{F}$-invariant projective line [LN1]. For a foliation of $\mathbf{C}P^1 \times \mathbf{C}P^1$ different from the two rulings (horizontal and vertical), there is a *bidegree*: $d_1(\mathcal{F}) := \sum_{p \in H} tang(H, \mathcal{F}, p)$, with H a horizontal line and $d_2(\mathcal{F}) := \sum_{p \in V} tang(V, \mathcal{F}, p)$, with V a vertical line.

Corollary 3.2.4 *If $\mathcal{F}$ is a foliation of $\mathbf{C}P^2$ with degree $d(\mathcal{F})$, then*

$$g(\mathcal{F}) = \frac{d(\mathcal{F})(d(\mathcal{F}) + 1)}{2} - \sum_{p \in Sing(\mathcal{F})} \delta_p(\mathcal{F}).$$

If $\mathcal{F}$ is a ruling of $\mathbf{C}P^1 \times \mathbf{C}P^1$, then $g(\mathcal{F}) = -1$. If $\mathcal{F}$ is a foliation of $\mathbf{C}P^1 \times \mathbf{C}P^1$ different from the rulings, with bidegree $(d_1(\mathcal{F}), d_2(\mathcal{F}))$, then

$$g(\mathcal{F}) = (d_1(\mathcal{F}) + 1)(d_2(\mathcal{F}) + 1) - \sum_{p \in Sing(\mathcal{F})} \delta_p(\mathcal{F}).$$

Proof We compute $p_a(\mathcal{F}) := \chi(\mathcal{O}_M) + \frac{1}{2}c_1(T^*_{\mathcal{F}}) \cdot c_1(N_{\mathcal{F}})$ in the cases of $\mathbf{C}P^2$ and $\mathbf{C}P^1 \times \mathbf{C}P^1$.

For a non $\mathcal{F}$-invariant line L of $\mathbf{C}P^2$, by §2.4

$$c_1(T^*_{\mathcal{F}}) \cdot L = tang(L, \mathcal{F}) - L \cdot L = d(\mathcal{F}) - 1$$

and

$$c_1(N_{\mathcal{F}}) \cdot L = tang(L, \mathcal{F}) + 2 - 2p_a(L) = tang(L, \mathcal{F}) + 2,$$

that is, $c_1(T^*_{\mathcal{F}}) = (d(\mathcal{F}) - 1)[L]$ and $c_1(N_{\mathcal{F}}) = (d(\mathcal{F}) + 2)[L]$, where $[L] \in H^2(\mathbf{C}P^2, \mathbb{Z})$ is the Poincaré dual class of L. Since $\chi(\mathcal{O}_{\mathbf{C}P^2}) = 1$ we obtain the formula for $\mathbf{C}P^2$.

In the case of $\mathbf{C}P^1 \times \mathbf{C}P^1$, when $\mathcal{F}$ is not a ruling, if H and V are respectively horizontal and vertical non $\mathcal{F}$-invariant lines , by analogous reasoning using $tang(H, \mathcal{F})$ and $tang(V, \mathcal{F})$, we conclude that

$$c_1(T^*_{\mathcal{F}}) = d_2(\mathcal{F})[H] + d_1(\mathcal{F})[V] \quad \text{and} \quad c_1(N_{\mathcal{F}}) = (d_2(\mathcal{F}) + 2)[H] + (d_1(\mathcal{F}) + 2)[V],$$

where $[H]$ and $[V]$ are the Poincaré dual classes of H and V in $H^2(\mathbf{C}P^1 \times \mathbf{C}P^1, \mathbb{Z})$, which proves the formula for $\mathbf{C}P^1 \times \mathbf{C}P^1$.

If $\mathcal{F}$ is a ruling, for example the vertical one, by §2.3 (7)

$$c_1(T^*_{\mathcal{F}}) \cdot V = Z(V, \mathcal{F}) + 2p_a(V) - 2 = -2$$

and

$$c_1(T^*_{\mathcal{F}}) \cdot H = tang(H, \mathcal{F}) - H \cdot H = 0,$$

hence $c_1(T^*_{\mathcal{F}}) = -2[H]$. Since

$$c_1(N_{\mathcal{F}}) \cdot H = tang(\mathcal{F}, H) + 2 = 2 \quad \text{and} \quad c_1(N_{\mathcal{F}}) \cdot V = Z(\mathcal{F}, V) + V \cdot V = 0$$

we conclude that $c_1(N_{\mathcal{F}}) = 2[V]$. This gives $g(\mathcal{F}) = 1 + \frac{1}{2}(-2[H] \cdot 2[V]) = -1$.

□

3.3 Examples

Example 3.3.1 Let $\mathcal{F}$ be a reduced foliation of $\mathbf{C}P^2$. Then

$$N_{\mathcal{F}} = (d(\mathcal{F}) + 2)\mathcal{O}(L),$$

where L is projective line L, and $N_{\mathcal{F}}$ is an ample line bundle of $\mathbf{C}P^2$. By Corollaries 3.1.4 and 3.2.4,

$$h^0(T^*_{\mathcal{F}}) = g(\mathcal{F}) = \frac{d(\mathcal{F})(d(\mathcal{F}) + 1)}{2}.$$

Example 3.3.2 Let $\mathcal{F}$ be the radial foliation of $\mathbf{C}P^2$, that is, the pencil of projective lines passing by a point p. A reduced foliation $\widetilde{\mathcal{F}}$ associated to the radial foliation is the regular ruling obtained from $\mathcal{F}$ by a blowing up at p. We denote by Σ_1 the resulting surface. Then

$$h^0(\Sigma_1, T^*_{\widetilde{\mathcal{F}}}) = 0, \quad h^1(\Sigma_1, T^*_{\widetilde{\mathcal{F}}}) = 1 \quad \text{and} \quad h^2(\Sigma_1, T^*_{\widetilde{\mathcal{F}}}) = 0.$$

In fact, if $h^0(\Sigma_1, T^*_{\widetilde{\mathcal{F}}}) > 0$ then there is a non-trivial holomorphic 1-form along $T_{\widetilde{\mathcal{F}}}$. But the ruling $\widetilde{\mathcal{F}}$ is locally analytically isomorphic to $\mathbf{C}P^1 \times \Delta$, with Δ a complex disc. We arrive at a contradiction with the fact that there is no non-trivial holomorphic 1-form along $T\mathbf{C}P^1$. By Corollary 3.1.3, $h^2(\Sigma_1, T^*_{\widetilde{\mathcal{F}}}) = 0$ and, by Corollary 3.2.4, $g(\mathcal{F}) = -1$, which gives $h^1(\Sigma_1, T^*_{\widetilde{\mathcal{F}}}) = 1$.

Example 3.3.3 Let $\mathcal{F}$ be a foliation in $\mathbf{C}P^2$ and $\mathcal{G}$ its strict transform by a *quadratic birational* transformation $Q : \mathbf{C}P^2 \to \mathbf{C}P^2$ [Be]. After change of homogeneous coordinates, Q is given by

$$Q(x_0 : x_1 : x_2) = (x_1x_2 : x_0x_2 : x_0x_1).$$

There is a factorization $Q = \sigma_2 \circ \sigma_1^{-1}$, where

1) $\sigma_1 : M \to \mathbf{C}P^2$ is given by one blowing up at at each vertex $p_0 = (1 : 0 : 0)$, $p_1 = (0 : 1 : 0)$ and $p_2 = (0 : 0 : 1)$ of the triangle $\Delta = \{x_0x_1x_2 = 0\}$ (E_i denoting the exceptional lines) and

2) $\sigma_2 : M \to \mathbf{C}P^2$ is the blowing-down of the strict transformass $\widetilde{L}_i$ by σ_1 of the sides $L_i = \{x_i = 0\}$ of Δ.

Let Δ' denote the projective triangle which is the strict transform by Q of Δ and p'_i denote its vertices (see *Figure* 1 below). Denote by $\mathcal{F}'$ the strict transform $\sigma_1^*(\mathcal{F})$ and $\mathcal{G} := (Q^{-1})^*(\mathcal{F}) = (\sigma_2^{-1})^*(\mathcal{F}')$.

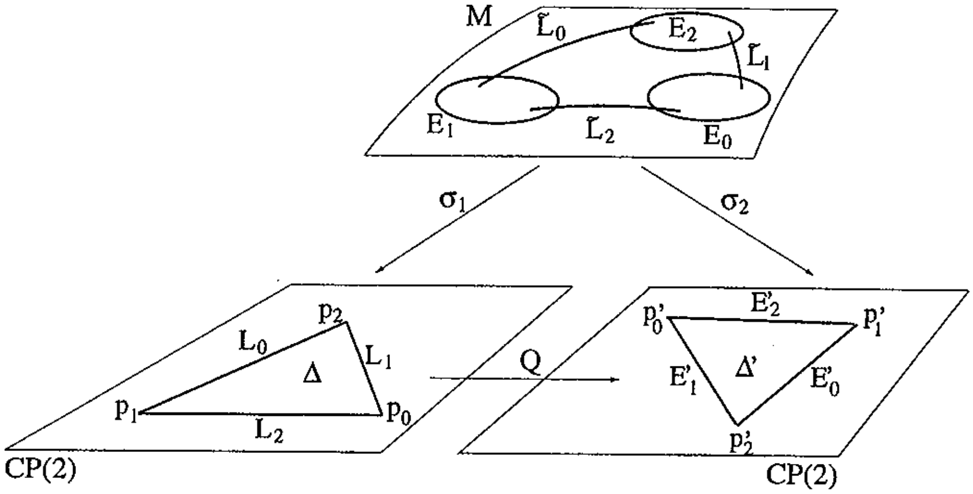

Figure 1: Quadratic birational transformation

Suppose that the sides L_i of Δ are non $\mathcal{F}$-invariant, $Sing(\mathcal{F}) \cap \Delta \subset \{p_0\}$ and L_i are chosen generically (in order that $\widetilde{L}_i \cap E_i$ are regular points of $\mathcal{F}'$). Then we assert:

a): $d(\mathcal{G}) = 2d(\mathcal{F}) + 2 - m_{p_0}$;

b): the singularities of $\mathcal{G}$ are of types:

i) isomorphic to singularities of $\mathcal{F}$ different of p_0,

ii) isomorphic to singularities at the exceptional line of one blowing up at p_0,

iii) two reduced singularities resulting from the blowing ups at the vertices p_1 and p_2 of Δ,

iv) three dicritical points p'_0, p'_1 and p'_2 without singularities at the exceptional line of one blowing up, obtained by blowing-down of $\widetilde{L}_i$;

c): $m_{p'_0} = d(\mathcal{F}) + 2$, $m_{p'_1} = m_{p'_2} = d(\mathcal{F}) + 2 - m_{p_0}$.

Assertions a), b) and c) imply that $g(\mathcal{G}) = g(\mathcal{F})$. In fact, by c):

$$\sum_{p \in Sing\mathcal{R}(\mathcal{G})} m_p(m_p - 1) = \sum_{i=0}^{2} m_{p'_i}(m_{p'_i} - 1) + \sum_{p \in Sing\mathcal{R}(\mathcal{G}) - \{p'_0, p'_1, p'_2\}} m_p(m_p - 1) =$$

$$= (d(\mathcal{F}) + 2)(d(\mathcal{F}) + 1) + 2(d(\mathcal{F}) + 2 - m_{p_0})(d(\mathcal{F}) + 1 - m_{p_0}) +$$

$$+ \sum_{p \in Sing\mathcal{R}(\mathcal{G}) - \{p'_0, p'_1, p'_2\}} m_p(m_p - 1).$$

By b):

$$\sum_{p \in Sing\mathcal{R}(\mathcal{G})} m_p(m_p - 1) = (d(\mathcal{F}) + 2)(d(\mathcal{F}) + 1) +$$

$$+2(d(\mathcal{F}) + 2 - m_{p_0})(d(\mathcal{F}) + 1 - m_{p_0}) + \sum_{p \in Sing\mathcal{R}(\mathcal{F}) - \{p_0\}} m_p(m_p - 1).$$

Using Corollary 3.2.4, the definition of $\delta_p(\mathcal{G})$ and item a), we obtain:

$$\begin{aligned} g(\mathcal{G}) &= \frac{1}{2}(2d(\mathcal{F})+2-m_{p_0})(2d(\mathcal{F})+3-m_{p_0}) - \\ &\quad -\frac{1}{2}[(d(\mathcal{F})+2)(d(\mathcal{F})+1)+2(d(\mathcal{F})+2-m_{p_0})(d(\mathcal{F})+1-m_{p_0})] - \\ &\quad -\frac{1}{2}\sum_{p\in Sing\mathcal{R}(\mathcal{F})-\{p_0\}} m_p(m_p-1) \\ &= \frac{1}{2}d(\mathcal{F})(d(\mathcal{F})+1)-\frac{1}{2}m_{p_0}(m_{p_0}-1)-\frac{1}{2}\sum_{p\in Sing\mathcal{R}(\mathcal{F})-\{p_0\}} m_p(m_p-1) \\ &= g(\mathcal{F}). \end{aligned}$$

Proof of the Assertions:

a): Take a smooth conic C containing p_0, p_1 and p_2 . If $\widetilde{C}$ denotes the strict transform of C by σ_1, then by §2.4

$$tang_{p_i}(C,\mathcal{F}) = tang_{\widetilde{C}\cap E_i}(\widetilde{C},\mathcal{F}') + m_{p_i}.$$

Then

$$\begin{aligned} tang(\widetilde{C},\mathcal{F}') &= tang(C,\mathcal{F}) - m_{p_0} \\ &= c_1(N_{\mathcal{F}})\cdot C + 2p_a(C) - 2 \\ &= d(C)(d(\mathcal{F})+2) - 2 - m_{p_0} \\ &= 2d(\mathcal{F}) + 2 - m_{p_0}. \end{aligned}$$

Remark that $\widetilde{C}$ does not intersect the strict transforms $\widetilde{L}_i$. Otherwise C would be tangent to some side of Δ and the contact between C and Δ would be greater than $6 = d(C).d(\Delta)$. Therefore the blowing-downs of $\widetilde{L}_i$ effected by σ_2 does not affect $\widetilde{C}$. The image of $\widetilde{C}$ by σ_2 is denoted L and $L = \sigma_2\circ\sigma_1^{-1}(C) = Q(C) \subset \mathbf{C}P^2$. The image by σ_2 of the exceptional lines E_i of σ_1 are projective lines denoted $E_i' \subset \mathbf{C}P^2$. Since $\sharp(L\cap E_i') = 1$ we conclude that L is also a projective line and that

$$\begin{aligned} d(\mathcal{G}) &:= tang(L,\mathcal{G}) \\ &= tang(\widetilde{C},\mathcal{F}') \\ &= 2d(\mathcal{F}) + 2 - m_{p_0}. \end{aligned}$$

b): follows from the description of Q and the hypotheses about Δ.

c): since p_0' is dicritical we have by §2.4 $m_{p_0'} = tang(L_0',\mathcal{F}')+2$, where L_0' is the exceptional line of the blowing up at p_0'. But L_0' is the strict transform of $L_0 := p_1p_2$. Since $tang(L_0,\mathcal{F}) = tang(L_0',\mathcal{F}')$, we conclude that $m_{p_0'} = d(\mathcal{F})+2$. For $m_{p_1'}$ and $m_{p_2'}$ we obtain, by the same reasoning applied to L_1 and L_2, $m_{p_1'} = m_{p_2'} = d(\mathcal{F}) + 2 - m_{p_0}$.

Let us consider, for a foliation $\mathcal{F}$ of a projective compact complex surface M, $\mathcal{B}(\mathcal{F})$ the class of foliations of M birationally equivalent to $\mathcal{F}$.

A *minimal* surface is a surface without embedded Riemann spheres E with $E.E = -1$ (that is, without exceptional lines of a blowing up).

We remark that, in the case M is a minimal non-ruled surface, $\mathcal{B}(\mathcal{F})$ is composed only by foliations isomorphic to $\mathcal{F}$. In fact, a birational transformation between minimal non-ruled surfaces is an isomorphism [Be].

Proposition 3.3.4 *Let M be a minimal ruled surface. Let $\mathcal{F}$ be a reduced foliation of M without invariant rational curves. If $\mathcal{G} \in \mathcal{B}(\mathcal{F})$, then either $p_a(\mathcal{G}) > g(\mathcal{F})$ or clG is isomorphic to $\mathcal{F}$.*

In the case of $\mathbf{C}P^2$, let us define the *degree* of $\mathcal{B}(\mathcal{F})$, denoted $d(\mathcal{B}(\mathcal{F}))$, as the minimum degree of the foliations in $\mathcal{B}(\mathcal{F})$. We recall that in each component of the space of foliations of $\mathbf{C}P^2$ with fixed degree $d \geq 2$, there is an open dense set $\mathcal{U}_d$ such that if $\mathcal{F} \in \mathcal{U}_d$, then $\mathcal{F}$ has no compact solution and has only reduced singularities [J] [LN1]. The previous proposition implies:

Corollary 3.3.5 *Let $\mathcal{F}$ be a reduced foliation in $\mathbf{C}P^2$ with $d(\mathcal{F}) \geq 2$ and without invariant rational curves. Then*
i) $d(\mathcal{B}(\mathcal{F})) = d(\mathcal{F})$ and
ii) if there is $\mathcal{G} \in \mathcal{B}(\mathcal{F})$ with $d(\mathcal{G}) = d(\mathcal{B}(\mathcal{F}))$, then $\mathcal{G}$ is isomorphic to $\mathcal{F}$.

Proof (Prop. 3.3.4) Let $\mathcal{G} = \phi^*(\mathcal{F})$ with $\phi : M \to M$ a birational transformation. Supposing $p_a(\mathcal{G}) \leq g(\mathcal{F})$, we will show that ϕ^{-1} cannot collapse any curve $C \subset M$ to a point, that is, ϕ is well defined in all M. Since M is minimal, it follows from the Structure Theorem that $\phi : M \to M$ is in fact an isomorphism [Be].

In fact, supposing $p_a(\mathcal{G}) \leq g(\mathcal{F})$, by Theorem 3.2.3,

$$g(\mathcal{F}) = g(\mathcal{G}) = p_a(\mathcal{G}) - \sum_{p \in Sing(\mathcal{G})} \delta_p(\mathcal{G})$$

and we conclude that $p_a(\mathcal{G}) = g(\mathcal{F})$ and $\delta_p(\mathcal{G}) = 0$ for every $p \in Sing(\mathcal{G})$. Suppose that ϕ^{-1} collapse a curve $C \subset M$ to a point p'.

By the Structure Theorem, some strict transform of C is an exceptional line and then C is a rational curve. Then by hypothesis C is non $\mathcal{F}$-invariant. Collapsing C by ϕ^{-1} gives rise to a singularity $p' \in Sing(\mathcal{G})$ with infinite number of local separatrices and a dicritical point $q \in Sing\mathcal{R}(\mathcal{G}, p')$, which has $m_q \geq 2$. Then $\delta_{p'}(\mathcal{G}) > 0$, a contradiction.

□

3.4 Topological invariance of $g(\mathcal{F})$

A foliation $\mathcal{F}$ of M is *topologically equivalent* to a foliation $\mathcal{G}$ of N if there is an orientation preserving homeomorphism $h : M \to N$, with $h(Sing(\mathcal{F})) = Sing(\mathcal{G})$ and h sending the leaves of $\mathcal{F}$ to the leaves of $\mathcal{G}$.

Topologically equivalent foliations of $\mathbf{C}P^2$ have the same degree. More generally, $c_1(T_{\mathcal{F}})$ is a topological invariant of a foliation $\mathcal{F}$ [GSV], that is, by the isomorphism h^* induced in cohomology, $c_1(T_{\mathcal{F}}) = h^* c_1(T_{\mathcal{G}})$.

A singularity of foliation is called *generalized curve* [CLNS] when there is no saddle-node in its resolution. A foliation $\mathcal{F}$ of a compact surface M will be called a generalized curve of M when every singularity of $\mathcal{F}$ is a generalized curve.

Theorem 3.4.1 *Let $\mathcal{F}$ be a generalized curve of M and $\mathcal{G}$ a foliation of N, with $c_2(N) = c_2(M)$. Then*
i) if there is a bijection between $Sing(\mathcal{G})$ and $Sing(\mathcal{F})$ and each singularity of G is topologically equivalent to a singularity of $\mathcal{F}$, then $\chi(\mathcal{G}) = \chi(\mathcal{F})$;
ii): if $\mathcal{G}$ is topologically equivalent to $\mathcal{F}$, then $g(\mathcal{G}) = g(\mathcal{F})$.

Proof If the surfaces M and N are homeomorphic, then $\chi(\mathcal{O}_N) = \chi(\mathcal{O}_M)$ [BPV]. The Milnor numbers $Det(\mathcal{F}, p)$ are topological invariants [CLNS]. A singularity topologically equivalent to a generalized curve is a generalized curve having isomorphic desingularization [CLNS]. Therefore itens i) and ii) follows from Theorem 3.2.3.

□

A *generic quasi-hyperbolic singularity* is defined in [MS] and is a generalization of the concept of generalized curve with finite number of separatrices.

An *analytic deformation* of a compact surface M_0 with base space $(\mathbb{C}^p, 0)$ is given by a proper analytic map $F : \mathcal{M} \to \Delta$ of maximal rank, where $\mathcal{M}$ is a $(2+p)$-dimensional complex manifold and $\Delta \subset \mathbb{C}^p$, such that $F^{-1}(0) = M_0$. The smooth compact surfaces $M_t = F^{-1}(t)$ are C^∞-diffeomorphic [BPV]. The family M_t will also be called an analytic deformation of M_0 with base space $\mathbb{C}^p$.

A *deformation* with base space $(\mathbb{C}^p, 0)$ of a singularity of a local foliation $\mathcal{G}_0$ represented by $w = a(x,y)dx + b(x,y)dy$ is given by a 1-form

$$\theta = A(x,y,t)dx + B(x,y,t)dy, \quad A, B \in \mathcal{O}_{(\mathbb{C}^{2+p},0)}$$

with

$$A(x,y,0) = a(x,y), \; B(x,y,0) = b(x,y).$$

Theorem **B** of [MS] asserts that topologically trivial deformations of generic quasi hyperbolic singularities have isomorphic desingularizations. Since under an analytic deformation $\{M_t\}_t$ of a surface M_0, the C^∞-structure is preserved, it is possible to prove a version of Theorem 3.4.1 for a deformation $\{\mathcal{F}_t\}_t$ of $\mathcal{F} = \mathcal{F}_0$ in $M = M_0$.

Theorem 3.4.2 *Let $\mathcal{F}_0$ be a foliation in M_0 and suppose that each singularity p of $\mathcal{F}_0$ is generic quasi hyperbolic. Let M_t be an analytic deformation of M_0 with base space $\mathbb{C}^p$. Consider a family of foliations $\{\mathcal{F}_t\}_t$ in M_t and suppose that the singularities of $\{\mathcal{F}_t\}_t$ are local deformations with base space $\mathbb{C}^p$ of the singularities of $\mathcal{G}_0$.*

If the local deformations of the singularities of $\mathcal{F}_0$ in $\{\mathcal{F}_t\}_t$ are topologically trivial, then $g(\mathcal{F}_t) = g(\mathcal{F}_0)$

4 Fibrations and pencils of curves

In this section, we consider foliations with rational first integral of *projective* compact complex surfaces, that is, pencils of curves of algebraic surfaces.

A fibration of a smooth (connected) complex surface M (not necessarily compact) over a smooth connected complex curve Δ is given by $f : M \to \Delta$ a proper, surjective, holomorphic map.

A point $p \in M$ is a critical point of a fibration $f : M \to \Delta$ if $df = 0$ at p.[1] A critical value at Δ is a point in the image by f of a critical point. By Remmert's Theorem, the set of critical values in Δ, denoted Σ, is discrete. The fibers $M_s := f^{-1}(s)$ with $s \in \Sigma$ are the *singular fibers*. By Ehresmann-Shih's Theorem,

$$f_{|f^{-1}(\Delta-\Sigma)} : f^{-1}(\Delta - \Sigma) \to \Delta - \Sigma$$

is C^∞ locally trivial. Hence the fibers $f^{-1}(t)$, for $t \in \Delta - \Sigma$, are all C^∞-diffeomorphic and any such fiber is called a generic fiber, denoted M_g.

[1] We call attention to the fact that, in this definition of fibration, the set of critical points may have complex dimension 1.

If a fibration f has connected generic fibers, then, by the fact that f is proper and by Sard's Theorem, also the singular fibers are connected. A fibration is connected when all its fibers are connected.

4.1 Elimination of base points

In order to reduce assertions about pencils of curves to assertions about fibrations, we consider the process of *elimination of base points.*

Suppose $\mathcal{F}$ in M has a rational first integral ϕ. Locally, ϕ can be represented by $\phi_i = \frac{f_i}{g_i}$, where $f_i, g_i \in \mathcal{O}(\mathcal{U}_i)$ and $\sharp\{f_i = 0\} \cap \{g_i = 0\} < \infty$. The points $\{f_i = 0\} \cap \{g_i = 0\}$ are the *base points* of $\mathcal{F}$ in $\mathcal{U}_i$ and correspond to the singular points of $\mathcal{F}$ with local meromorphic (non-holomorphic) first integral. There is a well-defined holomorphic map

$$\phi_{i|\,\mathcal{U}_i - (\{f_i=0\}\cap\{g_i=0\})} : \mathcal{U}_i - (\{f_i = 0\} \cap \{g_i = 0\}) \to \mathbf{C}P^1.$$

There is a finite sequence of blowing ups σ at the base-points of $\mathcal{F}$ and at the base-points of the transformed pencils, such that for $\sigma : \widetilde{M} \to M$ we obtain a well-defined map $\phi \circ \sigma : \widetilde{M} \to \mathbf{C}P^1$.

The description of the process at each base-point p, keeping the notation introduced above, is as follows.

Let $\sigma : M' \to M$ be a blowing up at $p \in \mathcal{U}_i$ with $E = \sigma^{-1}(p)$ and without loss of generality suppose $\nu_p(G_i) \geq \nu_p(F_i)$, where $F_i = \{f_i = 0\}$ and $G_i = \{g_i = 0\}$. We have $\sigma^*(F_i) = \widetilde{F_i} + \nu_p(F_i)E$ and $\sigma^*(G_i) = \widetilde{G_i} + \nu_p(G_i)E$, where $\widetilde{F_i}$ and $\widetilde{G_i}$ are the strict transforms of F_i and G_i respectively. Consider $F_i' := \widetilde{F_i}$ and $G_i' := \widetilde{G_i} + (\nu_p(G_i) - \nu_p(F_i))E$, denoting $F_i' = \{f_i' = 0\}$ and $G_i' = \{g_i' = 0\}$. Then the transformed pencil $\mathcal{F}'$ in M' has a rational first integral which restricted to $\sigma^{-1}(\mathcal{U}_i)$ is given by

$$\phi_i' = \frac{f_i'}{g_i'} : \sigma^{-1}(\mathcal{U}_i) \to \mathbf{C}P^1,$$

because $F_i' - G_i' = \sigma^*(F_i) - \sigma^*(G_i)$.

Also $F_i'.G_i' = \sigma^*(F_i).\sigma^*(G_i) - \nu_p^2(F_i) = F_i.G_i - \nu_p^2(F_i)$ and, after performing at most $F_i.G_i$ blowing ups at the base-points belonging to $\mathcal{U}_i$, we obtain a rational first integral in $\widetilde{M}$, which restricted to an open $\mathcal{V}$ is given by

$$\phi_i^{(n)} = \frac{f_i^{(n)}}{g_i^{(n)}} : \mathcal{V} \to \mathbf{C}P^1.$$

Since $\{f_i^{(n)} = 0\} \cap \{g_i^{(n)} = 0\} = \emptyset$, we obtain $\phi_i^{(n)} : \mathcal{V} \to \mathbf{C}P^1$ as a well-defined map and the transformed pencil $\widetilde{\mathcal{F}}$ in $\mathcal{V}$ is a *fibration.*

A curve C of a pencil is a *generic* curve (*critical* curve) if its strict transform $\widetilde{C}$ by an elimination of base points of $\mathcal{F}$ is a generic (singular) fiber of the fibration $\mathcal{F}'$ obtained.

A pencil has irreducible generic curve when the fibration obtained has connected generic fiber. We distinguish critical curves according if their strict transform are components of different singular fibers.

4.2 Inequalities between $g(\mathcal{F})$ and $g(C)$

In this section, if a curve $C = \sum_i n_i C_i$ (with C_i reduced and irreducible curves) has some multiple component, that is, some $n_i > 1$, then $(C)_{red} := \sum_i C_i$. The Milnor number of C at p is the Milnor number of $(C)_{red}$ at p.

A $(-n)$-curve is an embedded Riemann sphere with self-intersection number $-n$. A minimal curve is a curve without (-1)-curves as components, that is, a curve without components which can be contracted.

Lemma 4.2.1 *Let $\mathcal{F}$ be a connected fibration $f : M \to S$ of a compact surface M over a compact Riemann surface S, M_g and M_s denoting respectively generic and singular fibers. Let $\chi(M_s)$ denote the topological Euler characteristic of $(M_s)_{red}$. Then*

$$\chi(\mathcal{F}) - \sum_{p \in Sing(\mathcal{F})} 2\delta_p(\mathcal{F}) = \chi(S)\chi(M_g) + \sum_s (\chi(M_s) - \chi(M_g)) - Det(\mathcal{F}),$$

where the sum on s runs along all critical values of f in S.

Proof By Theorem 3.2.3 and the definition of $\chi(\mathcal{F})$

$$\chi(\mathcal{F}) = c_2(M) - Det(\mathcal{F}) + \sum_{p \in Sing(\mathcal{F})} 2\delta_p(\mathcal{F}).$$

By a well known fact about fibrations [BPV]

$$c_2(M) = \chi(S)\chi(M_g) + \sum_s (\chi(M_s) - \chi(M_g)).$$

□

Remark 1 The *Hirzebruch surfaces* Σ_n are characterized as analytic $\mathbf{C}P^1$-bundles over $\mathbf{C}P^1$ having an analytic section s_n with self-intersection number $s_n \cdot s_n = -n$ [Be].

Let h denote a fiber of a ruling of Σ_n. Consider one blowing up of a point $p \in h - s_n$. Then the strict transform of h, denoted $\widetilde{h}$, is a (-1)-curve and can be contracted. If $\widetilde{s_n}$ denote the strict transform of s_n under the contraction of $\widetilde{h}$, $\widetilde{s_n} \cdot \widetilde{s_n} = s_n \cdot s_n + 1 = -(n-1)$ and the surface obtained by the modification of Σ_n is Σ_{n-1}. Repeating the process we obtain Σ_1. The ruling of Σ_1 is the strict transform under a blowing up at a point $p \in \mathbf{C}P^2$ of the pencil of lines containing p.

Proposition 4.2.2 *Let $\mathcal{F}$ be a connected fibration given by $f : M \to \mathbf{C}P^1$ having generic fiber M_g with $\chi(M_g) = 2$. Then*

i) after a finite number of blowing-downs of M, the surface M' obtained is isomorphic to a Hirzebruch surface Σ_n and the modification $\mathcal{F}'$ of $\mathcal{F}$ is a ruling of Σ_n;

ii) $\chi(\mathcal{F}) = 4$.

Proof Assertion i) is a particular case of Proposition V.4.3 of [BPV]. By Lemma 4.2.1 applied to $\mathcal{F}'$, $\chi(\mathcal{F}') = \chi(\mathbf{C}P^1)\chi(M_g) = 4$ and $\chi(\mathcal{F}) = \chi(\mathcal{F}')$ because is a birational invariant of $\mathcal{F}$.

□

After elimination of base-points we obtain, using Remark 1 and Proposition 4.2.2:

Corollary 4.2.3 *Let $\mathcal{F}$ be a pencil of curves of a surface M having irreducible generic curve C with geometrical genus $g(C) = 0$. Then $\mathcal{F}$ is birationally equivalent to the pencil of lines in $\mathbf{C}P^2$ through a point, that is, the* radial *foliation in $\mathbf{C}P^2$.*

Proposition 4.2.4 *Let $\mathcal{F}$ be a connected fibration given by $f : M \to \mathbf{C}P^1$. If $\chi(\mathcal{F}) \leq -2$, then $\chi(M_g) \leq -2$.*

In particular, when M is a rational surface, $g(\mathcal{F}) \geq 2$ implies $g(M_g) \geq 2$.

Proof By Proposition 4.2.2 above, it is enough to show that $\chi(M_g) \neq 0$. Let us suppose, by absurd, that $\chi(M_g) = 0$. Applying Lemma 4.2.1 to $\mathcal{F}$ we obtain

$$\begin{aligned}\sum_s \chi(M_s) - Det(\mathcal{F}) &= \chi(\mathcal{F}) - \sum_{p \in Sing(\mathcal{F})} 2\delta_p(\mathcal{F}) \\ &\leq -2.\end{aligned}$$

But we assert that, in fact, $\sum_s \chi(M_s) - Det(\mathcal{F}) \geq 0$ and the contradiction proves that $\chi(M_g) \leq -2$. In order to prove the assertion, write

$$\sum_s \chi(M_s) - Det(\mathcal{F}) = \sum_s (\chi(M_s) - \sum_{p \in M_s} Det(\mathcal{F}, p)),$$

and consider, for each s, $f_s : \mathcal{U}_s \to \Delta$ a local elliptic fibration over a disc Δ having an unique singular fiber $M_s = f_s^{-1}(0)$. In this local situation, we consider two cases:
Case 1: $M_s = f_s^{-1}(0)$ is minimal. In this case, we can describe M_s using Kodaira's table of singular fibers of elliptic local fibrations [K][BPV].
Case 1.a): M_s is not a multiple fiber.

In the table below it is used the notation for singular fibers of [K], with $\chi(M_s)$ is the topological Euler characteristic. *Figure* 2 below shows the dual graphs of some singular fibers.

M_s	**Components**	**Singularities**	$\chi(M_s) - \sum_{p \in M_s} Det(\mathcal{F}, p)$
I_0	1 elliptic curve	non-singular	$0 - 0 = 0$
I_1	1 rational curve	one node	$1 - 1 = 0$
$I_{b \geq 2}$	cycle of b (-2)-curves	normal crossings	$b - b = 0$
II	1 rational curve	simple cusp of order 2	$2 - 2 = 0$
III	2 (-2)-curves	contact of order 2	$3 - 3 = 0$
IV	3 (-2)-curves	triple crossing	$4 - 4 = 0$
$I^*_{b \geq 0}$	chain of $b + 5$ (-2)-curves	nodal crossings	$b + 6 - (b + 4) = 2$
II^*	chain of 9 (-2)-curves	nodal crossings	$10 - 8 = 2$
III^*	chain of 8 (-2)-curves	nodal crossings	$9 - 7 = 2$
IV^*	chain of 7 (-2)-curves	nodal crossings	$8 - 6 = 2$

We conclude that $\chi(M_s) - \sum_{p \in M_s} Det(\mathcal{F}, p) \geq 0$, which proves the assertion in this case.
Case 1.b): If M_s is a multiple fiber, that is, $M_s = mF$ for some $m > 1$.

Then according to [K], $F \cong I_0$, I_1, or I_b and hence $\chi(M_s) - \sum_{p \in M_s} Det(\mathcal{F}, p) = 0$.
Case 2: $M_s = f_s^{-1}(0)$ has some (-1)-curve as a component.

In this case, M_s is obtained from a singular fiber I_0,...,IV^*) by a sequence of blowing ups. The numbers $\chi(M_s) - \sum_{p \in M_s} Det(\mathcal{F}, p)$ are invariant by blowing ups in all cases, except for fibers of type III and IV.

One blowing up at the triple point of IV increases in 2 the number

$$\chi(M_s) - \sum_{p \in M_s} Det(\mathcal{F}, p).$$

One blowing up at the singular point of III creates a triple point at the exceptional line and we are at a situation like that of the fiber IV.

□

After elimination of base-points, we obtain by Proposition 4.2.4,

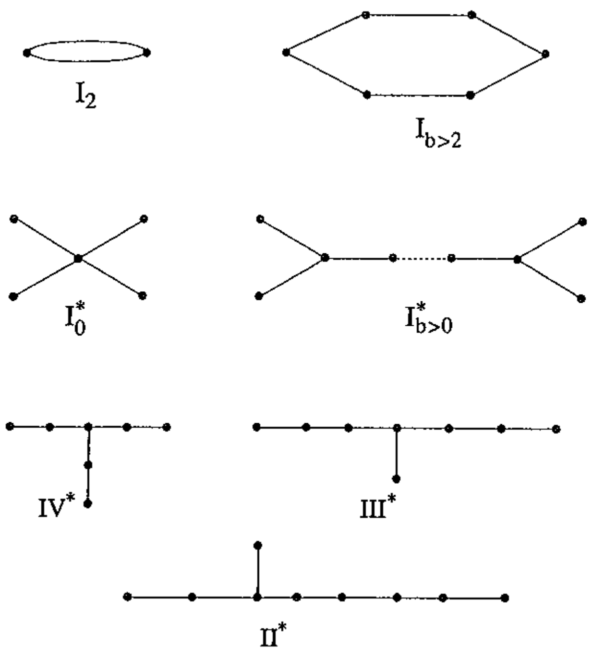

Figure 2: Dual graphs

Corollary 4.2.5 *Let $\mathcal{F}$ be a pencil of curves of a surface M, with irreducible generic curve C. If $\chi(\mathcal{F}) \leq -2$, then $\chi(C) \leq -2$, where $\chi(C)$ is the geometric Euler characteristic.*

The converse is not true (the pencils $\mathcal{F}_{k \geq 4}$ of Example 5.1.2 have $\chi(C) \leq -2$ and $\chi(\mathcal{F}_k) = 4$).

Example 4.2.6 In $\mathbf{C}P^2$ with homogeneous coordinates $(x_0 : x_1 : x_2)$, consider the elliptic pencil given by

$$f_1 - \tau f_2 = 0, \quad \tau \in \mathbf{C}P^1,$$

where $C_1 = \{f_1 = 0\}$ and $C_2 = \{f_2 = 0\}$ are cubic curves with transversal intersections. By Bertini's Theorem the generic cubic is smooth out of the base-points and by Bézout Theorem it is smooth at the base-points.

We will consider the different pencils obtained varying the curves C_1 and C_2. The notation for the pencils refers to the type of critical curve, according to Kodaira's Table in the proof of Proposition 4.2.4.

Let $\mathcal{F}_{I_3}$ be the pencil given by

$$x_0^3 + x_1^3 + x_2^3 - 3\tau x_0 x_1 x_2 = 0, \quad \tau \in \mathbf{C}P^1,$$

that is, with C_1 a smooth cubic and C_2 a triangle. Then a generic curve of $\mathcal{F}_{I_3}$ is a smooth cubic L, that is $g(L) = 1$, but it is well-known that, for each value $\tau \in \{1, e^{\frac{2\pi i}{3}}, e^{\frac{4\pi i}{3}}, \infty\}$, the corresponding curve of $\mathcal{F}_{I_3}$ is a projective triangle. $Sing(\mathcal{F}_{I_3})$ contains i): 9 radial points (with $Det(\mathcal{F}_{I_3}, p) = 1$ and $\delta_p(\mathcal{F}_{I_3}) = 1$) at the transversal intersection $C_1 \cap C_2$ and

ii): 12 nodal singularities of the 4 projective triangles (with $Det(\mathcal{F}_{I_3}, p) = 1$ and $\delta_p(\mathcal{F}_{I_3}) = 0$). We have $d(\mathcal{F}_{I_3}) = 4$ and along the singularities listed in i) and ii)

$$\sum_{p \in i), ii)} Det(\mathcal{F}_{I_3}, p) = 21 = 4^2 + 4 + 1,$$

wich imples that in fact these are all the singularities of $\mathcal{F}_{I_3}$ in $\mathbf{C}P^2$. By the formula for $\mathbf{C}P^2$, $g(\mathcal{F}_{I_3}) = \frac{4.5}{2} - \frac{2.9}{2} = 1$.

Let $\mathcal{F}_{I_1}$, $\mathcal{F}_{II}$, $\mathcal{F}_{I_2}$, $\mathcal{F}_{III}$ denote respectively the pencils obtained as above, when C_1 is a smooth cubic and C_2 is a) a nodal cubic, b) a cuspidal cubic, c) union of a smooth conic and a transversal line and d) union of a smooth conic and a tangent line. In all these cases, the formula for $\mathbf{C}P^2$ gives $g(\mathcal{F}) \leq 1$.

Let $\mathcal{F}_{IV}$ be the pencil formed when C_1 is smooth and C_2 is given as three concurrent lines. Then at the singularity p of $\{f_2 = 0\}$, $m_p(\mathcal{F}_{IV}) = \nu(\mathcal{F}_{IV}, p) = 2$ and the genus formula gives $g(\mathcal{F}_{IV}) \leq \frac{4.5}{2} - \frac{2.9}{2} - 1 = 0$.

Proposition 4.2.7 *Let $\mathcal{F}$ be a connected fibration given by $f : M \to S$ over S a compact Riemann surface, with generic fiber M_g.*

i) If all singular fibers of $\mathcal{F}$ are free from multiple components, then

$$\chi(\mathcal{F}) - \chi(S)\chi(M_g) = \sum_{p \in Sing(\mathcal{F})} 2\delta_p(\mathcal{F}).$$

ii) If $\chi(M_g) \leq -2$, all the singular fibers M_s are minimal and $(M_s)_{red}$ have at most nodal points, then

$$\chi(\mathcal{F}) - \chi(S)\chi(M_g) \geq K$$

where K denotes the number of singular fibers having some multiple component.

Moreover,

$$\chi(\mathcal{F}) - \chi(S)\chi(M_g) = 0$$

only if all singular fibers of $\mathcal{F}$ are free from multiple components.

In particular, for a connected fibration $\mathcal{F}$, given by $g : M \to \mathbf{C}P^1$, with M a rational surface, satisfying the hypotheses of ii) above

$$g(\mathcal{F}) \leq 2g(M_g) - \frac{K+2}{2}$$

(with equality holding only if the singular fibers are free from multiple components).

Example 4.2.8 Let $\mathcal{H}_k$ be the pencils generated by two smooth curves C_1 and C_2 intersecting transversally, with $d(C_1) = d(C_2) = k$, $\mathcal{H}_k$ having only reduced nodal critical curves and smooth generic curve C. Then the degree of $\mathcal{H}_k$ verifies $d(\mathcal{H}_k) = 2k - 2$ and since the non-reduced singularities are radial points ($\delta_p(\mathcal{H}_k) = 1$) at the intersection $C_1 \cap C_2$, we obtain

$$g(\mathcal{H}_k) = \frac{(2k-2)(2k-1)}{2} - k^2 = (k-1)(k-2) - 1 = 2g(C) - 1 > 2, \quad \text{if} \quad k \geq 4.$$

Before proving the Proposition 4.2.7, we state a consequence.

Corollary 4.2.9 *Let $\mathcal{F}$ be a pencil in a surface M with irreducible generic curve C and $\chi(C) \leq -2$. Suppose that each critical curve C_λ has no rational components and that $(C_\lambda)_{red}$ has at most nodal points. Let $J \geq 0$ be the number of critical curves having some multiple component. Then*

$$\chi(\mathcal{F}) - 2\chi(C) \geq J,$$

where $\chi(C)$ is the geometric Euler characteristic of a generic curve. In particular, for a rational surface M and a pencil $\mathcal{F}$ satisfying the hypotheses,

$$g(\mathcal{F}) \leq 2g(C) - \frac{J+2}{2}.$$

Proof The hypotheses that there are no rational components of critical curves and that each critical curve is at most nodal assure that it is possible to obtain, by resolution of singularities of $\mathcal{F}$, a fibration $\mathcal{F}'$ with minimal and nodal singular fibres. In fact, a minimal fiber of $\mathcal{F}'$ is non-nodal only if the strict transform of a rational component blown-down.

Since $\chi(C) \leq -2$, by Prop. 4.2.7-ii), $\chi(\mathcal{F}') - 2\chi(C) \geq K$, where K is the number of singular fibers with some multiple component.

Since there are no rational components of the critical curves of $\mathcal{F}$, $J \leq K$. In fact, a multiple componet of $\mathcal{F}$ does not appear in $\mathcal{F}'$ only if it is blown-down and then it is a rational component of a critical curve.

□

Proposition 4.2.7 follows immediately from Lemma 4.2.1 and the next Lemma:

Lemma 4.2.10 *Let $\mathcal{F}$ be a connected fibration in M, given by $f : M \to S$, with generic and singular fibers M_g and M_s respectively.*
i) If M_s is free from multiple components, then

$$\chi(M_s) - \chi(M_g) - \sum_{p \in M_s} Det(\mathcal{F}, p) = 0;$$

ii) If $\chi(M_g) \leq -2$, M_s is minimal and $(M_s)_{red}$ has at most nodal points, then

$$\chi(M_s) - \chi(M_g) - \sum_{p \in M_s} Det(\mathcal{F}, p) \geq 0,$$

and the equality holds only if M_s is free from multiple components.

We remark that assertion *ii)* of the lemma above is not true in general without the hypothesis $\chi(M_g) \leq -2$. For example, an elliptic fibration with a singular fiber of type $M_s = mI_0$, $m > 1$, i.e., M_s a smooth elliptic curve with multiplicity $m > 1$, has

$$\chi(M_s) - \chi(M_g) - \sum_{p \in M_s} Det(\mathcal{F}, p) = \chi(M_s) = 0.$$

Proof (Lemma 4.2.10)

Assertion *i)* is Iversen's Formula [Iv].

Proof of *ii)*: In [X] it is proved that, for any minimal singular fiber $M_s = \sum_{i=1}^{m} n_i C_i$ of a connected fibration with $\chi(M_g) \leq -2$,

$$\text{(a)} \qquad g(M_g) \geq b_{M_s} + \sum_{i=1}^{m} g(\widetilde{C_i})$$

with equality only if M_s is free from multiple components, where

$$b_{M_s} = \sum_{i=1}^{m} (p_a(C_i) - g(\widetilde{C_i})) + \frac{\sum_{i=1}^{m} (C_i \cdot \sum_{j \neq i}^{m} C_j)}{2} - (m-1) \geq 0,$$

and $g(\widetilde{C_i})$ is the genus of the normalization of C_i.

(In Appendix A we give a detailed proof of this result, for reading convenience.)

We show that this fact implies the result of ii), under the hypotheses of nodal singularities of $(M_s)_{red}$. In fact, by (a), $2(m-1) + 2 - 2g(M_g) \leq \sum_{i=1}^{m} (2 - 2g(\widetilde{C_i})) - 2b_{M_s}$ and

$$
\begin{aligned}
2(m-1)+\chi(M_g) \quad \le \quad & \sum_{i=1}^{m}\chi(\tilde{C}_i) - 2\sum_{i=1}^{m}(p_a(C_i) - g(\widetilde{C_i})) - \\
& -\sum_{i=1}^{m}(C_i \cdot \sum_{j\neq i}^{m} C_j) + 2(m-1),
\end{aligned}
$$

that is,

$$
\sum_{i=1}^{m}\chi(\tilde{C}_i) - 2\sum_{i=1}^{m}(p_a(C_i) - g(\widetilde{C_i})) - \sum_{i=1}^{m}(C_i \cdot \sum_{j\neq i}^{m} C_j) - \chi(M_g) \ge 0.
$$

By the hypotheses of nodal singularities of $(M_s)_{red}$, $\sum_{i=1}^{m}(p_a(C_i) - g(\widetilde{C_i}))$ is the number of singularities of the components C_i $(i = 1, ..., m)$ of M_s and $\frac{1}{2}\sum_{i=1}^{m}(C_i \cdot \sum_{j\neq i}^{m} C_j)$ is the number of intersection $C_i \cap C_j$ between all different components. Since at nodal points p, $Det(\mathcal{F}, p) = 1$, we obtain

$$
\sum_{i=1}^{m}\chi(\tilde{C}_i) - 2\sum_{i=1}^{m}(p_a(C_i) - g(\widetilde{C_i})) - \sum_{i=1}^{m}(C_i \cdot \sum_{j\neq i}^{m} C_j) = \sum_{i=1}^{m}\chi(\tilde{C}_i) - 2\sum_{p\in M_s} Det(\mathcal{F}, p)
$$

and $\chi(M_s) = \sum_{i=1}^{m}\chi(\widetilde{C_i}) - \sum_{p\in M_s} Det(\mathcal{F}, p)$, because normalization of C_i amounts to separate the local branches at nodal singularities.

□

Corollary 4.2.11 *Let $\mathcal{F}$ be a pencil in a projective surface M, with irreducible generic curve C. Suppose that:*

a) all curves of $\mathcal{F}$ are free from multiple components,

b) each singularity having infinite number of local separatrices is dicritical (the same supposition is made for singularities at each step of a resolution).

Then:

$$
\chi(\mathcal{F}) - 2\chi(C) = \sum_{p\notin Dicr\mathcal{R}(\mathcal{F})} m_p(m_p - 1)
$$

where $Dicr\mathcal{R}(\mathcal{F}) \subset Sing\mathcal{R}(\mathcal{F})$ is the set of all dicritical points.

Proof Consider an elimination of base-points of $\mathcal{F}$ such that only base-points (dicritical points) are blow-up. Denote $\mathcal{F}'$ the resulting fibration over $\mathbf{C}P^1$, given by $f' : M' \to \mathbf{C}P^1$.

By hypothesis b), no exceptional line introduced by the sequence of blowing ups defining M' is a fiber of $\mathcal{F}'$. This fact and hypothesis a) suffice to assert that all the fibers of $f' : M' \to \mathbf{C}P^1$ are free from multiple components. Lemmas 4.2.1 and 4.2.10.i) applied to $\mathcal{F}'$ complete the proof.

□

Corollary 4.2.12 *Let $\mathcal{F}$ be a foliation in a projective surface M with rational first integral $\phi = \frac{f_m}{f}$ and irreducible generic curve C. Suppose that*

i) $L = \{f = 0\}$ is smooth,

ii) $L_m = \{f_m = 0\}$ is the unique curve of $\mathcal{F}$ having some multiple component and

iii) $(L_m)_{red} \cap L$ is transversal. Then

$$
\chi(\mathcal{F}) - \chi(C) = \sum_{p\in Sing(\mathcal{F})} 2\delta_p(\mathcal{F}) + \chi(L_m) - 2C \cdot (L_m)_{red} - \sum_{p\in L_m - L} Det(\mathcal{F}, p)
$$

where $\chi(L_m)$ is the topological Euler characteristic and $\chi(C)$ is the geometric Euler characteristic.

Example 4.2.13 An example of foliation satisfying the hypotheses of Corollary 4.2.12. Let C be a curve of $\mathbf{C}P^2$ without multiple components, which is supposed transversal to the line at infinity L_∞. Let $p = 0$ be a reduced affine equation of C in $\mathbf{C}P^2 - L_\infty$. Denote $\mathcal{F}$ the foliation of $\mathbf{C}P^2$ which extends the foliation given by $dp = 0$. Then $d(\mathcal{F}) = d(C) - 1$ and $\mathcal{F}$ has as a rational first integral $\Phi = \frac{P(x_0:x_1:x_2)}{x_0^k}$, where $P = 0$ is a homogeneous equation of C, $L_\infty = \{x_0 = 0\}$ and $k := d(C)$.

Proof (Corollary 4.2.12) Let $L_m = \sum_{i=1}^k n_i C_i$, with C_i reduced and irreducible curves and $(L_m)_{red} := \sum_{i=1}^k C_i$. Being transversal the intersection $L \cap (L_m)_{red}$, each singularity (base-point) $p \in L \cap (L_m)_{red}$ of $\mathcal{F}$ belongs to exactly one component C_i of $(L_m)_{red}$. There are local coordinates (u, v) around p in which $\mathcal{F}$ is represented by

$$\omega = n_i u dv - v du = 0,$$

where $C_i = \{v = 0\}$ and $L = \{u = 0\}$.

Consider an elimination of the base-point $p \in L \cap C_i$, denoted σ_p, such that the blowing ups are only base points of the transformed pencils. The transformed foliation under the s-th blowing up σ_s in this sequence, $0 \leq s \leq n_i - 1$, is locally represented by:

$$\widetilde{\omega} = (n_i - s)\widetilde{u}d\widetilde{v} - \widetilde{v}d\widetilde{u} = 0$$

that is, $\sigma_p = \sigma_{n_i} \circ \ldots \circ \sigma_1$ is the elimination of the base-point $p \in L \cap C_i$ and the n_i-th blowing up σ_{n_i} is at a dicritical (radial) point:

$$\widetilde{\omega} = \widetilde{u}d\widetilde{v} - \widetilde{v}d\widetilde{u} = 0.$$

Let us denote by σ the simultaneous elimination of all base-points $p \in L \cap L_m$, $L_m = \sum_{i=1}^k n_i C_i$. We conclude that after $\sum_{i=1}^k n_i (L \cdot C_i)$ blowing ups we obtain from $\mathcal{F}$ a fibration $\mathcal{F}'$, given by $f = \phi \circ \sigma : M' \to \mathbf{C}P^1$. Denote L'_m the fiber of $\mathcal{F}'$ which is the union of a) the strict transform by σ of L_m with b) the components of the exceptional divisor of σ which are contained in fibers of $\mathcal{F}'$ (see *Figure* 3 below).

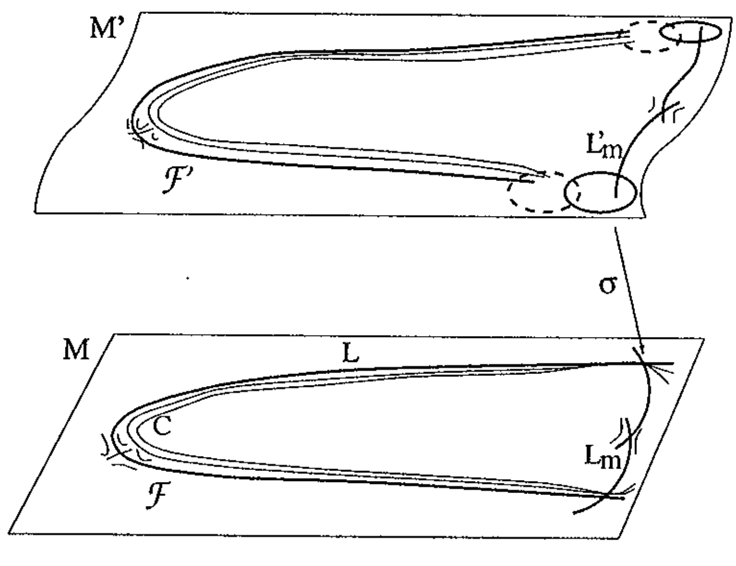

Figure 3: Elimination of base-points

Since the exceptional curve of the n_i-th blowing up is not contained in a fiber of $\mathcal{F}'$ the topological Euler characteristics $\chi(L'_m)$ and $\chi(L_m)$ are related by

$$\chi(L'_m) = \chi(L_m) + \sum_{i=1}^{k}(n_i - 1)(L \cdot C_i).$$

By the same reason

$$\sum_{p \in L'_m} Det(\mathcal{F}', p) = \sum_{p \in L_m - L} Det(\mathcal{F}, p) + \sum_{i=1}^{k}(n_i - 1)(L \cdot C_i)$$

and hence

$$\chi(L'_m) - \sum_{p \in L'_m} Det(\mathcal{F}', p) = \chi(L_m) - \sum_{p \in L_m - L} Det(\mathcal{F}, p).$$

By Lemma 4.2.1:

$$\chi(\mathcal{F}') - \sum_{p \in Sing(\mathcal{F}')} 2\delta_p(\mathcal{F}') = c_2(M') - Det(\mathcal{F}')$$

and denoting L_s the singular fibers of $\mathcal{F}'$ different of L'_m:

$$\chi(\mathcal{F}') - \sum_{p \in Sing(\mathcal{F}')} 2\delta_p(\mathcal{F}') = 2\chi(L) + \sum_{s}(\chi(L_s) - \chi(L)) + \chi(L'_m) - \chi(L) - Det(\mathcal{F}').$$

Since by hypothesis the singular fibers $L_s \neq L'_m$ are free from multiple components, by Lemma 4.2.10.i):

$$\begin{aligned}
\chi(\mathcal{F}') - \sum_{p \in Sing(\mathcal{F}')} 2\delta_p(\mathcal{F}') &= \chi(L) + \chi(L'_m) - \sum_{p \in L'_m} Det(\mathcal{F}', p) \\
&= \chi(L) + \chi(L_m) - \sum_{p \in L_m - L} Det(\mathcal{F}, p)
\end{aligned}$$

and we conclude

$$\begin{aligned}
\chi(\mathcal{F}) - \sum_{p \in Sing(\mathcal{F})} 2\delta_p(\mathcal{F}) &= \chi(\mathcal{F}') - \sum_{p \in Sing(\mathcal{F}')} 2\delta_p(\mathcal{F}') - 2L \cdot (L_m)_{red} \\
&= \chi(L) + \chi(L_m) - 2L \cdot (L_m)_{red} - \sum_{p \in L_m - L} Det(\mathcal{F}, p).
\end{aligned}$$

In order to complete the proof, remark that by Bertini Theorem the possible singularities of a generic curve C are situated at $L \cap L_m$. But since

$$\sum_{p \in L \cap L_m} \nu_p(L, C) \geq \sum_{i=1}^{k} n_i C_i \cdot C = L_m \cdot C = C \cdot C,$$

the conclusion is that C is smooth at $L_m \cap L$, that is, $\chi(C) = \chi(L)$.

□

Remark 2 A fiber M_t of a fibration is stable [BPV] when i) M_t is free from multiple components, ii) M_t has at most nodal singularities and iii) M_t is minimal. A fibration is stable when all the fibers are stable.

Let $\mathcal{F}$ given by $f : M \to S$ be a connected fibration over a compact Riemann surface S. Let $\sigma : \widetilde{M} \to M$ be a sequence of blowing ups such that the fibers of the fibration $\widetilde{\mathcal{F}}$ given by $f \circ \sigma : \widetilde{M} \to M$ have as sets at most nodal singularities.

Given a ramified covering $\delta : T \to S$, T a compact Riemann surface, consider the pullback of $\widetilde{M}$ by δ, denoted $T \times_S \widetilde{M}$. This is in general a singular surface with non-isolated singularities. After normalization, $T \times_S \widetilde{M}$ has an unique (up to isomorphism) minimal desingularization denoted $\widetilde{N}$. A fibration $\widetilde{\mathcal{G}}$, given by $\widetilde{g} : \widetilde{N} \to T$, is obtained by the pullback of $\widetilde{\mathcal{F}}$ by δ to $T \times_S \widetilde{M}$ followed by the desingularization $\Phi : \widetilde{N} \to T \times_S \widetilde{M}$(see *Figure* 4 below)..

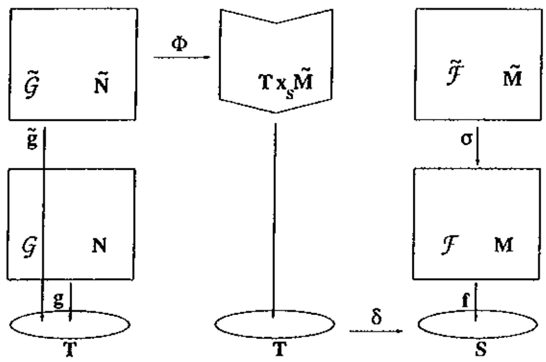

Figure 4: Stable reduction

The Global Stable Reduction Theorem [BPV] asserts that there exists a cyclic covering $\delta : T \to S$, ramified over the critical values of f and one extra point, such that the fibration $\widetilde{\mathcal{G}}$ of $\widetilde{N}$ has only singular fibers with nodal points and free from multiple components. After blowing-down of components of the fibers of $\widetilde{\mathcal{G}}$, we obtain a fibration $\mathcal{G}$ with stable fibers, $\mathcal{G}$ is a global stable reduction of $\mathcal{F}$.

By construction, the generic fiber of $\mathcal{G}$ is C^∞-diffeomorphic to the generic fiber of $\mathcal{F}$. By Lemma 4.2.1 and Lemma 4.2.10-i), $\chi(\mathcal{G}) = \chi(T)\chi(M_g)$.

4.3 An inequality for $g(\mathcal{F})$

Proposition 4.3.1 *Let $\mathcal{F}$ be a pencil of a surface M with irreducible generic curve. Then for any reduced foliation $\widetilde{\mathcal{F}}$ associated to $\mathcal{F}$ by a resolution $\mathcal{R}(\mathcal{F})$,*

$$g(\mathcal{F}) \leq \chi(\mathcal{O}_M) - \frac{1}{2}c_1(N_{\mathcal{F}}) \cdot c_1(M) + \frac{1}{2} \sum_{p \in Sing\mathcal{R}(\mathcal{F}) - Sing(\widetilde{\mathcal{F}})} m_p.$$

In particular, for $M = \mathbf{C}P^2$

$$g(\mathcal{F}) \leq 1 - \frac{3}{2}(d(\mathcal{F}) + 2)) + \frac{1}{2} \sum_{p \in Sing\mathcal{R}(\mathcal{F}) - Sing(\widetilde{\mathcal{F}})} m_p.$$

If all singularities of $\mathcal{F}$ in M are either a) of Morse type, i.e. represented by $\omega = xdy + ydx$

or b) dicritical points without singularities along the exceptional line of a blowing up, then

$$g(\mathcal{F}) = \chi(\mathcal{O}_M) - \frac{1}{2}c_1(N_{\mathcal{F}}) \cdot c_1(M) + \frac{1}{2} \sum_{p \in Dicr(\mathcal{F})} m_p,$$

where $Dicr(\mathcal{F})$ denotes the set of dicritical points of $\mathcal{F}$. In the case $M = \mathbf{C}P^2$:

$$g(\mathcal{F}) = 1 - \frac{3}{2}(d(\mathcal{F}) + 2)) + \frac{1}{2} \sum_{p \in Dicr(\mathcal{F})} m_p.$$

Remark 3 Each foliation $\mathcal{F}$ in $\mathbf{C}P^2$ is birationally equivalent to a foliation $\mathcal{F}'$ in $\mathbf{C}P^2$ with $Sing(\mathcal{F}')$ composed by either a) reduced singularities or b) dicritical points without singularities along the exceptional line of a blowing up [Ca1].

Remark 4 We can compare the result of Prop. 4.3.1 with the formula for the geometric genus of an irreducible curve $C \subset \mathbf{C}P^2$, when C is a generic curve of pencil $\mathcal{F}$ of $\mathbf{C}P^2$.

Let $\widetilde{\mathcal{F}}$ in M be a reduced fibration associated to a pencil $\mathcal{F}$ by means of a resolution $\mathcal{R}(\mathcal{F})$, given as a sequence of blowing ups $\sigma : M \to \mathbf{C}P^2$. For the strict transform $\widetilde{C}$ of C by by σ, it holds

$$d(C)^2 - \sum_{p \in Sing\mathcal{R}(\mathcal{F})} \nu_p^2 = \widetilde{C} \cdot \widetilde{C} = 0,$$

(where ν_p are the algebraic multiplicities of C and of its strict transforms and E_p are exceptional lines), because $\widetilde{C}$ is a generic fiber of $\widetilde{\mathcal{F}}$.

Then the genus formula

$$g(C) = \frac{1}{2}((d(C) - 1)(d(C) - 2) - \sum_{p \in Sing\mathcal{R}(\mathcal{F})} \nu_p(\nu_p - 1))$$

simplifies to

$$g(C) = 1 - \frac{3}{2}d(C) + \sum_{p \in Sing\mathcal{R}(\mathcal{F})} \frac{\nu_p}{2}.$$

Proof (Prop. 4.3.1) Consider a resolution of singularities $\mathcal{R}(\mathcal{F})$ (which, in particular, is an elimination of base-points) and the connected fibration $\widetilde{\mathcal{F}}$ obtained.

Suppose that S_1,...,S_k are all the singular fibers of the fibration $\widetilde{\mathcal{F}}$ obtained. Let $I(S_i, \widetilde{\mathcal{F}}, p)$ denote the index of [Su](which generalizes the Camacho-Sad indices [CS], [LN1]). According to the formula of sum of indices [Su]:

$$I(S_i, \widetilde{\mathcal{F}}) := \sum_{p \in S_i} I(S_i, \widetilde{\mathcal{F}}, p) = (S_i)_{red} \cdot (S_i)_{red}.$$

By Proposition 9 of [Br2], for fibrations,

$$I(S_i, \widetilde{\mathcal{F}}, p) = Tr(\widetilde{\mathcal{F}}, p)$$

(the Baum-Bott index of §2.3) and therefore:

$$\begin{aligned} c_1^2(N_{\widetilde{\mathcal{F}}}) &= \sum_{p \in Sing(\widetilde{\mathcal{F}})} Tr(\widetilde{\mathcal{F}}, p) \\ &= \sum_{i=1}^{k} (S_i)_{red} \cdot (S_i)_{red} \\ &\leq 0, \end{aligned}$$

where the inequality follows from Zariski Lemma for fibrations (Lemma 7.0.5, Appendix A).

According to §2.2 and §2.3,

$$c_1^2(N_{\widetilde{\mathcal{F}}}) = c_1^2(N_{\mathcal{F}}) - \sum_{p\in Sing\mathcal{R}(\mathcal{F})-Sing(\widetilde{\mathcal{F}})} m_p^2,$$

that is,

$$c_1^2(N_{\mathcal{F}}) - \sum_{p\in Sing\mathcal{R}(\mathcal{F})-Sing(\widetilde{\mathcal{F}})} m_p^2 \leq 0.$$

Since the singularities $p \in Sing(\widetilde{\mathcal{F}})$ have $m_p = 0, 1$,

$$\begin{aligned} g(\mathcal{F}) &= \chi(\mathcal{O}_M) + \frac{1}{2}(c_1(T^*_{\mathcal{F}})\cdot c_1(N_{\mathcal{F}}) - \sum_{p\in Sing\mathcal{R}(\mathcal{F})-Sing(\widetilde{\mathcal{F}})} m_p(m_p-1)) \\ &= \chi(\mathcal{O}_M) + \frac{1}{2}(c_1^2(N_{\mathcal{F}}) - c_1(N_{\mathcal{F}})\cdot c_1(M) - \sum_{p\in Sing\mathcal{R}(\mathcal{F})-Sing(\widetilde{\mathcal{F}})} m_p(m_p-1)) \\ &\leq \chi(\mathcal{O}_M) - \frac{1}{2}c_1(N_{\mathcal{F}})\cdot c_1(M) + \frac{1}{2}\sum_{p\in Sing\mathcal{R}(\mathcal{F})-Sing(\widetilde{\mathcal{F}})} m_p. \end{aligned}$$

Let us suppose now that the singular set of $\mathcal{F}$ is composed by Morse type singularities and dicritical points without singularities after a blowing up.

Let $\widetilde{\mathcal{F}}$ be the fibration obtained by a blowing up at each dicritical point of $\mathcal{F}$. By the hypotheses on the singularities, $\widetilde{\mathcal{F}}$ is a reduced foliation. At a Morse type singularity of $\widetilde{\mathcal{F}}$, $I(S_i, \widetilde{\mathcal{F}}, p) = 0$ and hence $I(S_i, \widetilde{\mathcal{F}}) = 0$. We conclude that $c_1^2(N_{\widetilde{\mathcal{F}}}) = 0$ and since $Dicr(\mathcal{F}) = Sing\mathcal{R}(\mathcal{F}) - Sing(\widetilde{\mathcal{F}})$, as above we prove

$$g(\mathcal{F}) = \chi(\mathcal{O}_M) - \frac{1}{2}c_1(N_{\mathcal{F}})\cdot c_1(M) + \frac{1}{2}\sum_{p\in Dicr(\mathcal{F})} m_p.$$

□

4.4 Poincaré problem for pencils

In [P] H. Poincaré proposed the problem of bounding the degree of the generic curve of a pencil $\mathcal{F}$ of CP^2 in terms of the degree $d(\mathcal{F})$ of the pencil. In general the problem is bounding the degree of an invariant curve in terms of the degree of the foliation. This general problem has no positive answer without hypotheses either on the singularities of the curve [CeLN], [CaC] or on the singularities of the foliation [CaC].

In order to precise the problem for pencils, consider $\mathcal{F}$ a pencil in $\mathbf{C}P^2$ with rational first integral $\Psi = \frac{F}{G}$, where F, G are homogeneous polynomials with the same degree d, $\gcd(F, G) = 1$. By the Second Bertini Theorem [J], the supposition that a generic curve $C = \{\tau F + \mu G = 0\}$ of the pencil is reducible is equivalent to the existence of polynomials F', G' with $d(F') = d(G') < d$ and a rational function l verifying

$$\frac{F}{G} = l(\frac{F'}{G'}).$$

Obviously, a bound to the degree $d(C)$ of the generic curve implies by the genus formula for plane curves the existence of a bound to its geometrical genus $g(C)$. In general, however, the existence of a bound to the geometrical genus $g(C)$ does not imply a solution to the Poincaré problem for pencils, as can be seen in the next example.

Example 4.4.1 Consider the pencils $\mathcal{F}_{pq}$ in $\mathbf{C}P^2$ extending the foliation given in affine coordinates (x, y) by $\omega = pxdy + qydx$, $\frac{p}{q} \in \mathbb{Q}^+$, gcd $(p, q) = 1$. Then there are rational first integrals

$$\Psi(x_0 : x_1 : x_2) = \frac{x_1^p x_2^q}{x_0^{p+q}},$$

extending the holomorphic first integral $\psi(x, y) = x^p y^q$. This means that the generic curves C_{pq} of $\mathcal{F}_{pq}$ have geometrical genus $g(C_{pq}) = 0$. The condition gcd $(p, q) = 1$ assures that C_{pq} is irreducible. For all $\mathcal{F}_{pq}$ remark that $d(\mathcal{F}_{pq}) = 1$ and $d(C_{pq}) = p + q$. Also remark that $g(\mathcal{F}_{pq}) = -1$, by Corollary4.2.3.

Example 4.4.2 Suppose $\mathcal{F}$ is a pencil in $\mathbf{C}P^2$ with irreducible generic curve C and suppose that the singularities having local meromorphic first integral are of type: $\omega = \lambda_p xdy - ydx + \omega_2$, $\lambda_p \in \mathbb{N}$. Recall a formula found in [P] and [Pa]:

$$g(C) = 1 + \frac{1}{2} \sum_{p \in Sing(\mathcal{F})} [(\lambda_p + 1)(1 - \frac{3}{d(\mathcal{F}) + 2}) - 1]\nu_p(C).$$

wich implies that if $d(\mathcal{F}) > 4$ then $g(C) \geq 2$. Moreover, if $d(\mathcal{F}) > 4$ and $g(C) \geq 2$ are given, then the formula implies that there is an upper bound to $\sum_{p \in Sing(\mathcal{F})}(\lambda_p + 1)\nu_p(C)$ and, since

$$g(C) = 1 - \frac{3}{2}d(C) + \frac{1}{2} \sum_{p \in Sing(\mathcal{F})} \lambda_p \nu_p(C)),$$

that there is an upper bound to $d(C)$, as remarked in [Pa].

The next result shows that, under certain conditions, it is possible to give an upper bound to the degree of the generic curve of a pencil $d(C)$ in terms of a): data of the pencil ($d(\mathcal{F})$ and $\chi(\mathcal{F})$) and b): the geometrical genus $g(C)$.

Theorem 4.4.3 *Let $\mathcal{F}$ be a pencil of curves in $\mathbf{C}P^2$ with irreducible generic curve C. Suppose that each critical curve L_λ has no rational components and $(L_\lambda)_{red}$ has at most nodal singularities. If $g(C) \geq 2$, then*

$$d(C) \leq \frac{1}{2}(d(\mathcal{F}) + 2) + \frac{1}{2}((d(\mathcal{F}) + 2)(\chi(\mathcal{F}) - 2\chi(C))(5 - 3\chi(C))).$$

Remark 5 About Theo. 4.4.3.

1. By Corollary 4.2.5, $g(\mathcal{F}) \geq 2$ implies $g(C) \geq 2$.

1. It is not supposed that unions of critical curves are nodal curves.

2. The hypothesis that the critical curves have as sets at most nodal singularities does not imply that the generic curves are nodal (see Example 4.4.1).

3. Let $\mathcal{F}$ be a pencil in $\mathbf{C}P^2$ with $g(C) \geq 2$ satisfying the hypotheses of Corolllary 4.2.11 and that the singular points with holomorphic first integral are reduced. Then $\chi(\mathcal{F}) - 2\chi(C) = 0$ and $d(C) \leq \frac{1}{2}(d(\mathcal{F}) + 2)$ by Theorem 4.4.3.

The next Lemma 4.4.4 will be used in the proof of the Theorem 4.4.3.

Remark 6 Consider $f : M \to S$ a connected fibration with generic fiber of genus $g \geq 0$. Let M_s be a singular fiber and suppose that $M_s = mD$, for $m \geq 1$, D a divisor. Then m divides $g-1$. In fact,

$$\begin{aligned} g = g(M_g) = p_a(M_g) = p_a(M_s) &:= 1 + \frac{1}{2}(M_s \cdot M_s + M_s \cdot K_M) \\ &= 1 + \frac{1}{2}(mD \cdot K_M), \end{aligned}$$

since $M_s \cdot M_s = 0$. That is, $D \cdot K_M = \frac{2g-2}{m}$. Since $D \cdot D = 0$, we conclude that

$$p_a(D) := 1 + \frac{1}{2} D \cdot K_M = 1 + \frac{g-1}{m},$$

as asserted.

This remark shows that there is no multiple fibers of connected rational fibrations (the case $g = 0$) and that there is no restriction on the multiplicity m in the case of elliptic connected fibrations (we recall again the fibers mI_0). In the case $g = 2$, for example, we have $m = 1$.

The next lemma (a particular case of Proposition 2 of [X]), generalizes the remark:

Lemma 4.4.4 *Let $f : M \to S$ be a connected fibration with generic fiber of genus $g \geq 2$. Suppose $M_s = \sum_i m_i C_i$ is a minimal singular fiber of f. Then for all i, $m_i \leq 6g$.*

We recall that a foliation in $\mathbf{C}P^2$ can be represented by a homogeneous 1-form

$$\Omega = \sum_{i=0}^{2} F_i(x_0 : x_1 : x_2) dx_i \quad \text{with} \quad \gcd(F_0, F_1, F_2) = 1$$

with the $F_i \in \mathbb{C}[x_0, x_1, x_2]$ homogeneous with the same degree $d = d(F_i)$, satisfying the condition $\sum_{i=0}^{2} x_i F_i = 0$. As it is well known $d(\mathcal{F}) = deg(\Omega) - 1 := d - 1$.

Lemma 4.4.5 *(Darboux) [J]*

Let $\mathcal{F}$ be a foliation of $\mathbf{C}P^2$ with rational first integral $\Psi = \frac{F}{G}$, with irreducible generic curve $C = \{\tau F + \mu G = 0\}$. Denote $L_\lambda = \sum_{\lambda i} L_{\lambda i}$ a critical curve (with $L_{\lambda i} = \{u_{\lambda i} = 0\}$ reduced and irreducible). Then

$$GdF - FdG = f\,\Omega \quad \textit{with} \quad f = \prod_{\lambda i} u_{\lambda 1}^{\alpha_{\lambda 1}-1} \dots u_{\lambda t}^{\alpha_{\lambda t}-1},$$

where the product runs for all critical curves and their components and Ω is a homogeneous polynomial 1-form with isolated singularities representing $\mathcal{F}$.

Proof (Theorem 4.4.3)

If $\mathcal{F}$ has irreducible generic curve $C = \{\tau F + \mu G = 0\}$, with the notation of Darboux's Lemma 4.4.5,

$$GdF - FdG = f\,\Omega \quad \text{with} \quad f = \prod_{\lambda i} u_{\lambda 1}^{\alpha_{\lambda 1}-1} \dots u_{\lambda t}^{\alpha_{\lambda t}-1}.$$

Taking degrees,

$$\begin{aligned} 2d(C) - 1 &= deg(\Omega) + \sum_{\lambda i} d(L_{\lambda i})(\alpha_i - 1) \\ &= d(\mathcal{F}) + 1 + \sum_{\lambda i} d(L_{\lambda i})(\alpha_i - 1). \end{aligned}$$

Hence

$$d(C) = \frac{1}{2}(d(\mathcal{F})+2) + \frac{1}{2}(\sum_{\lambda i} d(L_{\lambda i})(\alpha_i - 1))$$

and to give an upper bound to $d(C)$ is equivalent to prove that there are upper bounds to

a) the number of critical curves L_λ with some multiple component,

b) the degrees $d((L_\lambda)_{red})$ (where $(L_\lambda)_{red} = \sum_i L_{\lambda i}$) and

c) the multiplicities $\alpha_{\lambda i}$ of the multiple components of all L_λ.

About a): by Corollary 4.2.9 the number J of critical curves of $\mathcal{F}$ having some multiple component verifies

$$J \leq \chi(\mathcal{F}) - 2\chi(C);$$

b): for each λ, by [CeLN]

$$d((L_\lambda)_{red}) \leq d(\mathcal{F}) + 2;$$

c): By the proof of Corollary 4.2.9, there exists a fibration $\mathcal{F}'$ obtained by blowing ups of $\mathcal{F}$, having only minimal fibers M_s and $(M_s)_{red}$ with at most nodal singularities.

Then the multiplicities of the components of M_s are bounded by $6g(C)$ by Lemma 4.4.4. The transforms of curves L_λ are components of M_s and therefore $\alpha_{\lambda i} - 1 \leq 6g(C) - 1 = 5 - 3\chi(C)$. The theorem is proved.

□

For a curve L of $\mathbf{C}P^1 \times \mathbf{C}P^1$, $d_1(L) := L \cdot H$ and $d_2(L) := L \cdot V$, where H and V are respectively lines of the horizontal and vertical rulings.

Corollary 4.4.6 *(of Theorem 4.4.3) Let $\mathcal{F}$ be a pencil of curves in $\mathbf{C}P^1 \times \mathbf{C}P^1$ in the same hypotheses and notations of Th. 4.4.3. If L is a generic curve of $\mathcal{F}$, then*

$$\begin{aligned} d_1(L) + d_2(L) \;\; &\leq \;\; \frac{1}{2}(d_1(\mathcal{F}) + d_2(\mathcal{F}) + 4) + \\ &+\frac{1}{2}((d_1(\mathcal{F}) + d_2(\mathcal{F}) + 4)(\chi(\mathcal{F}) - 2\chi(L))(5 - 3\chi(L))). \end{aligned}$$

Proof (of Corollary 4.4.6) A standard birational transformation T from $\mathbf{C}P^1 \times \mathbf{C}P^1$ to $\mathbf{C}P^2$ is obtained as follows. Let $p \in \mathbf{C}P^1 \times \mathbf{C}P^1$ and denote H and V the horizontal and vertical lines of $\mathbf{C}P^1 \times \mathbf{C}P^1$ with $p = H \cap V$. Consider a blowing up at p and the strict transforms of H and V, denoted respectively, $\tilde{H}$ and $\tilde{V}$. Since $\tilde{H}$ and $\tilde{V}$ are (-1)-curves, we can blow-down these curves and the resulting surface is $\mathbf{C}P^2$. This standard modification is denoted $T : \mathbf{C}P^1 \times \mathbf{C}P^1 \to \mathbf{C}P^2$. The strict transforms of the lines of the horizontal an vertical rulings compose two pencils of projective lines passing by the two points $T(H)$ and $T(V)$. Then the strict transform L' of L by T^{-1} contains $T(H)$ and $T(V)$ and has $d(L') = d_1(L) + d_2(L)$.

Given $\mathcal{F}$, take a generic point p such that for H and V containing p, $Sing(\mathcal{F}) \cap (H \cup V) = \emptyset$ and $tang(H, \mathcal{F}, p) = tang(V, \mathcal{F}, p) = 0$. Then the strict transform $\mathcal{F}'$ in $\mathbf{C}P^2$ of $\mathcal{F}$ by T^{-1} verifies

$$d(\mathcal{F}') = d_1(\mathcal{F}) + d_2(\mathcal{F}) + 2.$$

In fact, if S denotes the strict transform of a generic horizontal line H_ϵ of $\mathbf{C}P^1 \times \mathbf{C}P^1$, then S is a projective line non $\mathcal{F}'$-invariant and by §2.4:

$$\begin{aligned} d(\mathcal{F}') &= tang(H_\epsilon, \mathcal{F}) + tang(S, \mathcal{F}', T(V)) \\ &= d_1(\mathcal{F}) + m_{T(V)}(\mathcal{F}') \\ &= d_1(\mathcal{F}) + d_2(\mathcal{F}) + 2. \end{aligned}$$

Applying Th. 4.4.3 to $\mathcal{F}'$ and L' and using the birational invariance of $\chi(\mathcal{F})$ and $\chi(L)$ we prove the assertion. □

5 Foliations with negative $g(\mathcal{F})$

5.1 Examples

Example 5.1.1 Let $M = C \times L$ be the product of two compact Riemann surfaces C and L. Consider the fibrations $\mathcal{F}_1$ and $\mathcal{F}_2$ given by projections $p_1 : M \to C$ and $p_2 : M \to L$. Then, by Lemma 4.2.1, $\chi(\mathcal{F}_i) = \chi(C)\chi(L)$. As it is known [Be]

$$\begin{aligned} \chi(\mathcal{O}_M) &= \frac{1}{12}(c_1^2(M) + c_2(M)) \\ &= (g(C) - 1)(g(L) - 1) \end{aligned}$$

and then $g(\mathcal{F}_)) = -(g(C)-1)(g(L)-1)$, although $\mathcal{F}_1$ and $\mathcal{F}_2$ are not birationally equivalent if $g(C) \neq g(L)$.

When M is the rational surface $M = \mathbf{C}P^1 \times \mathbf{C}P^1$ we obtain $g(\mathcal{F}) = -1$ and as already remarked in §4.2, this fibration is birationally equivalent to the radial foliation of $\mathbf{C}P^2$.

When M is the ruled surface $M = C \times \mathbf{C}P^1$ with $g(C) \geq 1$, then $g(\mathcal{F}) \geq 0$.

When $M = C \times L$, with $g(L) \geq 2$ and $g(C) \geq 2$, then M is a surface of general type in Enriques-Kodaira classification [Be]. In this case $g(\mathcal{F}) \leq -1$ can be made arbitrarily negative. We recall that ruled (non-rational) surfaces and general type surfaces are not birationally equivalent to $\mathbf{C}P^2$.

Remark 7 We recall a geometric interpretation of the critical curves, which will be useful for the examples.

Let M be a projective surface and $|C|$ a very ample complete linear system of curves. If $M = \mathbf{C}P^2$, $|C| = |C_d|$ is the linear system of all curves with degree d.

By a singular curve we mean a curve which is singular as a set or has some multiple component. The geometric locus $\mathcal{D} \subset |C|$ of singular curves is the *discriminant hypersurface*. In the case $M = \mathbf{C}P^2$ and $|C| = |C_d|$, the degree of $\mathcal{D}_d$ in $|C_d| \cong \mathbf{C}P^{\frac{d(d+3)}{2}}$ is

$$deg(\mathcal{D}_d) = 3(d-1)^2$$

and a singular plane curve C_α of $|C_d|$ corresponds to a point in the hypersurface $\mathcal{D}_d$ with algebraic multiplicity

$$\nu_{C_\alpha}(\mathcal{D}_d) = [3(d(C_\alpha) - 1) - d^{(1)}]d^{(1)} + \sum_{p \in C_\alpha} \mu((C_\alpha)_{red}, p),$$

where $d^{(1)} := d(C_\alpha) - d((C_\alpha)_{red})$ and $\mu((C_\alpha)_{red}, p)$ is the Milnor number [AC].

A generic point of $\mathcal{D}$ corresponds therefore to a reduced curve of $|C|$ with the most generic type of singularity, that is one nodal point.

A subspace $\mathcal{P} \cong \mathbf{C}P^1$ of $|C_d|$, $\mathcal{P} \not\subset \mathcal{D}_d$ verifies $\mathcal{P}.\mathcal{D}_d = 3(d-1)^2$. For this reason, we expect, for example, that a pencil $\mathcal{F}$ in $\mathbf{C}P^2$ composed by generically smooth cubic curves will have 12 reduced nodal curves, provided $\mathcal{F}$ is generic in the sense that $\mathcal{P}_{\mathcal{F}} \cap \mathcal{D}_d$ is transversal in $|C_d|$.

By the formulas above, the multiple components of a curve can be seen as non-isolated singularities of the curve.

Example 5.1.2 Consider $k \geq 2$ different concurrent projective lines L_i. Consider the foliation $\mathcal{F}_k$ in $\mathbf{C}P^2$ extending $d(l_1 \ldots l_k) = 0$, where $l_i = 0$ is the affine reduced equation of L_i in $\mathbf{C}P^2 - L_\infty$.

We assert that a): the generic curve of $\mathcal{F}_k$ is smooth; b) $d(\mathcal{F}_k) = k-1$ and c): $Sing(\mathcal{F}_k)$ is composed exactly by the point $p = \cap_{i=1}^k L_i$ and k singularities at the infinity $q_i \in L_\infty \cap (\cup_{i=1}^k L_i)$.

If b) and c) hold, then $g(\mathcal{F}) = -1$ (Corollary 3.2.4). If a) holds, then the generic curve has $g(C) = \frac{1}{2}(k-1)(k-2)$ and we conclude that the pencils $\mathcal{F}_k$ are pairwise non birationally equivalent. $\mathcal{F}_2$ is birationally equivalent to the radial foliation by Proposition 5.2.1 and the elliptic pencil $\mathcal{F}_3$ with $g(\mathcal{F}_3) = -1$ is non birationally equivalent to the elliptic pencils $\mathcal{F}$ of Example 4.2.6, which have $g(\mathcal{F}) = 1, 0$.

In order to prove a), we remark that if a generic curve of $\mathcal{F}_k$ has singularities, then by Bertini Theorem they must be at the points q_i at the infinity, which are the base-points of the pencil $\mathcal{F}_k$. But the contact of each (smooth) local branch of a $\mathcal{F}_k$-invariant curve with each line L_i at q_i is of order k: the local equations around q_i is $kudv - vdu = 0$. If the generic curve is singular at some q_i, then its contact with the curve $\cup_{i=1}^k L_i$ is greater than k^2, contradicting Bézout's Theorem.

In order to prove b), remark that by Darboux Lemma 4.4.5 it is equivalent to prove that the curves of $\mathcal{F}_k$ different of $C_\alpha := kL_\infty$ have no multiple component. Assertion a) is equivalent to

$$\mathcal{P} = \mathcal{P}_{\mathcal{F}_k} \not\subset \mathcal{D}_k,$$

where $\mathcal{D}_k \subset |C_k|$ is the discriminant hypersurface introduced in Remark 7. For $C_\alpha := kL_\infty$ as a point of $\mathcal{D}_k$:

$$\begin{aligned}\nu_{C_\alpha}(\mathcal{D}_k) &= [3(k-1) - (k-1)](k-1) + \sum_{p\ C_\alpha} \mu((C_\alpha)_{red}, p) \\ &= 2(k-1)^2.\end{aligned}$$

For $C_\beta := \sum_{i=1}^k L_i$, $\nu_{C_\beta}(\mathcal{D}_k) = \mu((C_\beta)_{red}, p) = (k-1)^2$. Since

$$3(k-1)^2 = d(\mathcal{D}_k) = \mathcal{P}_{\mathcal{F}_k}.\mathcal{D}_k \geq \nu_{C_\alpha}(\mathcal{D}_k) + \nu_{C_\beta}(\mathcal{D}_k) = 3(k-1)^2,$$

we conclude that in $\mathcal{F}_k$ there is no curve different of C_α with some multiple component. Also we have proved that there is no curve in $\mathcal{F}_k$ singular as set different from C_β and therefore assertion c) is proved.

Remark 8 With the notation of the previous Example 5.1.2, we assert that $\widetilde{\mathcal{F}}_k$, the strict transform of $\mathcal{F}_k$ by a blowing up at $p = \cap_{i=1}^k L_i$, is a *Riccati foliation* of the Hirzebruch surface Σ_1.

In fact, since $m_p = \nu_p(\mathcal{F}_k) = k-1 = d(\mathcal{F}_k)$, for any projective line $L \neq L_i$, $tang(L, \mathcal{F}_k, p) = d(\mathcal{F}_k)$. If Ω is the pencil of lines through $p = \cap_{i=1}^k L_i$, then the transformed foliation $\widetilde{\Omega}$ by blowing up at p is isomorphic to the ruling of Σ_1.

The transformed foliation $\widetilde{\mathcal{F}}_k$ is transversal to the generic lines $\widetilde{L}$ of the ruling $\widetilde{\Omega}$, because $tang(\widetilde{\mathcal{F}}_k, \widetilde{L}) = tang(\mathcal{F}_k, L) - m_p = 0$. A foliation transversal to generic lines of the ruling of Σ_1 is a Riccati foliation [LN2].

5.2 On birational classification

If $\mathcal{F}$ is a foliation of a projective surface M with $g(\mathcal{F}) < 0$, then, for any reduced associated foliation $\widetilde{\mathcal{F}}$ of $\widetilde{M}$, the divisor associated to $N_{\widetilde{\mathcal{F}}}$ is not ample in $\widetilde{M}$. This is the content of Corollary 3.1.4 and expresses a non-genericity of the condition $g(\mathcal{F}) < 0$.

By the formula for $\mathbf{C}P^2$ of Corollary 3.2.4, a foliation $\mathcal{F}$ in the plane with $g(\mathcal{F}) < 0$ must have many degenerate singularities. This implies, by the results of [J][LN1], that such foliations must be non generic elements in the space of foliations of fixed degree d, if $d \geq 2$.

If $\mathcal{F}$ is a pencil of plane curves, $g(\mathcal{F})$ gives information about the singularities of its generic and critical curves. If there are many isolated singularities of the curves of $\mathcal{F}$, it is expected an increasing of the sum $\sum_{p \in Sing(\mathcal{F})} \delta_p(\mathcal{F})$. If there are multiple components of critical curves, (that is, non-isolated singularities according to Remark 7), then $d(\mathcal{F})$ decreases by Darboux's Lemma 4.4.5. In both cases decreases

$$g(\mathcal{F}) = \frac{d(\mathcal{F})(d(\mathcal{F})+1)}{2} - \sum_{p \in Sing(\mathcal{F})} \delta_p(\mathcal{F}).$$

Nevertheless Example 5.1.2 shows that $g(\mathcal{F})$ is not sufficient for a birational characterization of pencils in $\mathbf{C}P^2$, even when $g(\mathcal{F}) < 0$.

Proposition 5.2.1 *Let $\mathcal{F}$ be a generalized curve in $\mathbf{C}P^2$ with $g(\mathcal{F}) < 0$ and $d(\mathcal{F}) = 1$. Then $\mathcal{F}$ is birationally equivalent to the radial foliation in $\mathbf{C}P^2$ and $g(\mathcal{F}) = -1$.*

The assertion is false for $d(\mathcal{F}) \geq 2$ (see Example 5.1.2).

The proof of Proposition 5.2.1 is based on the following Lemma:

Lemma 5.2.2 *Let $\mathcal{F}$ be a foliation in M with $\chi(\mathcal{F}) > c_2(M)$. Then there exists some singularity of $\mathcal{F}$ with infinite number of local separatrices.*

In particular, a foliation in $\mathbf{C}P^2$ with $g(\mathcal{F}) < 0$ has a point in $\mathbf{C}P^2$ with infinite number of local separatrices.

Proof By the hypothesis and Theorem 3.2.3

$$\begin{aligned} 0 &> c_2(M) - \chi(\mathcal{F}) \\ &= Det(\mathcal{F}) - \sum_{p \in Sing\mathcal{R}(\mathcal{F})} m_p(m_p - 1). \end{aligned}$$

Suppose, by absurd, that each singularity of $\mathcal{F}$ in M has a finite number of local separatrices. We will show that this implies that

$$Det(\mathcal{F}) - \sum_{p \in Sing\mathcal{R}(\mathcal{F})} m_p(m_p - 1) \geq 0,$$

a contradiction which proves the lemma.

If $\widetilde{\mathcal{F}}$ is a reduced foliation associated to $\mathcal{F}$ and $\sharp\mathcal{R}(\mathcal{F})$ is the number of blowing ups of the resolution $\mathcal{R}(\mathcal{F})$, then by §2.3 (6)

$$\begin{aligned} Det(\mathcal{F}) - \sum_{p \in Sing\mathcal{R}(\mathcal{F})} m_p(m_p - 1) &= Det(\widetilde{\mathcal{F}}) - \sharp\mathcal{R}(\mathcal{F}) \\ &\geq \sharp Sing(\widetilde{\mathcal{F}}) - \sharp\mathcal{R}(\mathcal{F}), \end{aligned}$$

and we will show that $\sharp Sing(\widetilde{\mathcal{F}}) \geq \sharp\mathcal{R}(\mathcal{F})$.

For each singularity p of $\mathcal{F}$, $\sharp\mathcal{R}(\mathcal{F},p)$ denotes the number of blowing ups in its resolution. Each component of the exceptional divisor D_p is $\widetilde{\mathcal{F}}$-invariant and D_p has $\sharp\mathcal{R}(\mathcal{F},p) - 1$ points of normal crossings. But by [CS], $\widetilde{\mathcal{F}}$ has at least one singularity along D_p which is not at the (normal) intersections of its components, that is, $\sharp Sing(\widetilde{\mathcal{F}}) \geq \sharp\mathcal{R}(\mathcal{F})$. □

Proof (Proposition 5.2.1) Each singularity p of $\mathcal{F}$ in $\mathbf{C}P^2$ has $m_p = 1$. In fact, if $m_p \geq 2$, then each line passing by p would have $tang(L, \mathcal{F}, p) \geq 2$ cf. §2.4 and therefore would be $\mathcal{F}$-invariant. $\mathcal{F}$ would be isomorphic to the radial foliation $\mathcal{R}$ and $d(\mathcal{R}) = 0$, contradiction.

Then each singularity of $\mathcal{F}$ in the plane, by the Jordan Canonical Form, belongs to one of the following types:

$a) : \omega = (x+y)dy - ydx + \omega_2 = 0,$

$b) : \omega = ydy + \omega_2 = 0,$

$c) : \omega = \lambda xdy - \mu ydx + \omega_2 = 0, \quad \lambda\mu \neq 0, \quad \frac{\lambda}{\mu} \neq 1, \quad \gcd(\lambda, \mu) = 1$

where $p = (0,0)$ and ω_2 a 1-form with zero of order at least two at $(0,0)$.

Type $a)$ generates by a blowing up a saddle-node and therefore is excluded.

Since $Det(\mathcal{F}) = 3$ §2.3 (3), if a point p is of type b), then it is in fact of type

$$b') \quad \omega = y(1 + A(x,y))dy + x^n(1 + B(x,y))dx = 0,$$

$$A(0,0) = B(0,0) = 0 \quad \text{and} \quad n = 2, 3.$$

By Lemma 5.2.2 there exists at least one singular point p_1 in the plane with infinite number of separatrices. Such point must be of type c) with $\frac{\lambda}{\mu} \in \mathbb{Q}^+ - \{1\}$ (non radial). Since $Det(\mathcal{F}, p_1) = 1$, there are other two (counted with multiplicities) singular points p_2, p_3 of $\mathcal{F}$ in the plane. We will show that both belong to type c).

Suppose, by absurd, that for instance p_2 is of type b'). Then $Det(\mathcal{F}, p_2) \geq 2$ and we conclude that in fact $Det(\mathcal{F}, p_2) = 2$ (that is, $p_2 = p_3$). Then $\mathcal{F}$ is represented at p_2 by:

$$b'') \quad \omega = y(1 + A(x,y))dy + x^2(1 + B(x,y))dx.$$

Let L_{12} denote the line joining p_1 and p_2. We conclude, as above, that L_{12} is $\mathcal{F}$-invariant. But analyzing the resolution of p_2 we see that at p_2 there is an unique irreducible local separatrix γ of type: $\gamma = \{x^3 - y^2 + h.o.t. = 0\}$. This contradiction shows that p_2 is not of type b').

Hence there are singularities $p_2 \neq p_3$ both of type c). We assert that p_1, p_2 and p_3 cannot be collinear. In fact, suppose L_{123} is the projective line containing p_1, p_2 and p_3. Consider $Z(L_{123}, \mathcal{F})$ the sum of Poincaré-Hopf indices along L_{123}. Clearly $Z(L_{123}, \mathcal{F}) \geq 3$, but by §2.3 (7)

$$Z(L_{123}, \mathcal{F}) + 2p_a(L_{123}) - 2 = c_1(T^*_{\mathcal{F}}) \cdot L_{123} = (d(\mathcal{F}) - 1) = 0,$$

that is, $Z(L_{123}, \mathcal{F}) = 2$. Hence we consider the $\mathcal{F}$-invariant projective triangle $L_{12} \cup L_{13} \cup L_{23}$, where $p_i, p_j \in L_{ij}$.

Case 1): $g(\mathcal{F}) < -1$.

The points p_1, p_2, p_3 are of type c) above, that is, each p_i $i = 1,2,3$ is represented by

$$\omega_i = \lambda_i xdy - \mu_i ydx + \omega_{i2} = 0, \quad \lambda_i\mu_i \neq 0, \quad \frac{\lambda_i}{\mu_i} \neq 1, \quad gcd(\lambda_i, \mu_i) = 1.$$

The local separatrices γ_i at p_i of type $\gamma_i = \{x^{\lambda_i} - ty^{\mu_i} + h.o.t = 0\}$, with $t \in \mathbb{C}^*$, are irreducible because $gcd(\lambda_i, \mu_i) = 1$. Hence in the resolution of each p_i it appears at most one dicritical (radial) point. Since we suppose

$$g(\mathcal{F}) = 1 - \sum_{q \in Sing\mathcal{R}(\mathcal{F})} \frac{m_q(m_q - 1)}{2} < -1$$

it must appear one radial point in the resolution of each p_i $i = 1,2,3$. Hence each p_i is represented by $\omega_i = \lambda_i xdy - \mu_i ydx + \omega_{i2} = 0$, with $\frac{\lambda_i}{\mu_i} \in \mathbb{Q}^+ - \{1\}$. The Camacho-Sad indices

[CS] verify $i_{p_i}(L_{ij},\mathcal{F}) = \frac{\lambda_i}{\mu_i}$ and $i_{p_i}(L_{ik},\mathcal{F}) = \frac{\mu_i}{\lambda_i}$ and since $\{p_1,p_2,p_3\} = Sing(\mathcal{F}) \cap (L_{12} \cup L_{13} \cup L_{23})$ we conclude that for some line L_{ij}

$$\sum_{p\in Sing(\mathcal{F})\cap L_{ij}} i_p(L_{ij},\mathcal{F}) > 1$$

contradicting the formula of sum of indices, which asserts that

$$\sum_{p\in Sing(\mathcal{F})\cap L_{ij}} i_p(L_{ij},\mathcal{F}) = L_{ij}\cdot L_{ij} = 1.$$

The contradiction shows that the case $g(\mathcal{F}) < -1$ does not occur.
Case 2): $g(\mathcal{F}) = -1$

We conclude there are exactly two dicritical (radial) points in the resolution of $\mathcal{F}$. Therefore we can assert that among the points p_1, p_2 and p_3 of type c) there are two points p_1 and p_2 represented respectively by

$$\omega = \lambda_1 xdy - \mu_1 ydx + \omega_{12}, \quad \gcd(\lambda_1,\mu_1) = 1, \quad \frac{\lambda_1}{\mu_1} \in \mathbb{Q}^+ - \{1\},$$

$$\omega = \lambda_2 xdy - \mu_2 ydx + \omega_{22}, \quad \gcd(\lambda_2,\mu_2) = 1, \quad \frac{\lambda_2}{\mu_2} \in \mathbb{Q}^+ - \{1\}$$

and p_3 is represented by

$$\omega = \lambda_3 xdy - \mu_3 ydx + \omega_{32}, \quad \gcd(\lambda_3,\mu_3) = 1, \quad \frac{\lambda_3}{\mu_3} \in \mathbb{Q}^-.$$

Let $L_{12} := p_1.p_2$ be the line at infinity of $\mathbf{C}P^2$ and (z,w) affine coordinates of $\mathbb{C}^2 \cong \mathbf{C}P^2 - L_{12}$ in which $L_{13} = \{z=0\}$ and $L_{23} = \{w=0\}$. Then $\mathcal{F}$ is represented by a polynomial 1-form α of degree 1 [LN1]:

$$\alpha = (az+bw)dw - (cz+ew)dz, \quad a,b,c,e \in \mathbb{C}$$

The $\mathcal{F}$-invariance of $\{zw=0\}$ implies that b=c=0 and $\alpha = azdw - ewdz$, with $\frac{a}{e} = \frac{\lambda_3}{\mu_3} \in \mathbb{Q}^-$. Therefore $\psi(z,w) = z^{\lambda_3}w^{-\mu_3}$ is a holomorphic first integral for $\mathcal{F}$ in $\mathbb{C}^2$, extending to $\mathbf{C}P^2$ as a rational first integral $\Psi(x_0 : x_1 : x_2) = \frac{x_1^{\lambda_3}x_2^{-\mu_3}}{x_0^{\lambda_3-\mu_3}}$. Since $\gcd(\lambda_3,\mu_3) = 1$, then $\mathcal{F}$ is a pencil of irreducible generic rational curves in $\mathbf{C}P^2$ and therefore $\mathcal{F}$ is birationally equivalent to the radial foliation (Proposition 4.2.3).

$\square$

5.3 Conjectures

Question: Are pencils the unique type of foliations in $\mathbf{C}P^2$ *with* $g(\mathcal{F}) < 0$*?*
Conjecture 1: Let $\mathcal{F}$ *be a foliation of* $\mathbf{C}P^2$*. Then* $g(\mathcal{F}) \geq -1$*.*
Conjecture 2: Let $\mathcal{G}$ *be a fibration with rational connected generic fiber, given by* $g : M \to \mathbf{C}P^1$*. Suppose that the singular fibers of* $\mathcal{G}$ *have as sets at most nodal singularities. Then for any reduced foliation* $\mathcal{F}$ *of* M *,*

$$Det(\mathcal{F}) \geq Det(\mathcal{G}).$$

Assertion: Conjecture 2 implies Conjecture 1.

In fact, let M be the surface obtained from $\mathbf{C}P^2$ by a finite sequence of blowing ups σ defining a resolution of $\mathcal{F}$. Let $\widetilde{\mathcal{F}}$ in M be the reduced foliation associated to $\mathcal{F}$. Let $\mathcal{G}$ given by $g : M \to \mathbf{C}P^1$ denote the rational fibration in M which is the strict transform by σ of a radial foliation in $\mathbf{C}P^2$. By Theorem 3.2.3

$$c_2(M) = \chi(\mathcal{G}) + Det(\mathcal{G}) = \chi(\widetilde{\mathcal{F}}) + Det(\widetilde{\mathcal{F}}).$$

Supposing Conjecture 2, then $Det(\widetilde{\mathcal{F}}) \geq Det(\mathcal{G})$ and by Lemma 4.2.2

$$2 - 2g(\mathcal{F}) = \chi(\mathcal{F}) = \chi(\widetilde{\mathcal{F}}) \leq \chi(\mathcal{G}) = 4,$$

that is, $g(\mathcal{F}) \geq -1$.

6 Ramified coverings

A polynomial relation $P(y', x, y) = 0$, $y' = \frac{dy}{dx}$ defines a multiform differential equation that can be interpreted as a differential equation in a surface $P(w, x, y) = 0$, which is a ramified covering over the plane (x, y). This is a motivation to the study of ramified coverings and the variation of $\chi(\mathcal{F})$ under pullback of $\mathcal{F}$ by the projection of a ramified covering. After preliminaries on singular ramified coverings, we state in §6.4 a formula for the variation of $\chi(\mathcal{F})$ under pullbacks by generically finite maps.

6.1 2-Fold singular coverings

Let C be a curve free from multiple components of a non-singular surface M. Suppose there is a line bundle L on M such that

$$\mathcal{O}(C) = L \otimes L.$$

If C is given in $\mathcal{U}_i$ by $C = \{f_i = 0\}$ and $\psi_{ij} \in \mathcal{O}^*(\mathcal{U}_i \cap \mathcal{U}_j)$ is a transition function for L, then $\frac{f_i}{f_j} = \psi_{ij}^2$. We can suppose that $\mathcal{U}_i$ trivializes L and z_i is a complex coordinate for the fibers of $L_{|\mathcal{U}_i}$. Then the local equations $z_i^2 - f_i = 0$ define an analytic set of the total space of L denoted $M(C)$. In fact $M(C)$ is well defined, because in $\mathcal{U}_i \cap \mathcal{U}_j$ we have:

$$\begin{aligned} z_i^2 - f_i &= \psi_{ij}^2 z_j^2 - \psi_{ij}^2 f_j \\ &= \psi_{ij}^2 (z_j^2 - f_j). \end{aligned}$$

By definition $M(C)$ is a 2-*fold covering* of M ramified along C, with projection $\pi : M(C) \to M$. The singularities of $M(C)$ are the points which are singularities of C and are isolated because C is free from multiple components.

In the particular case $M = \mathbf{C}P^2$, the hypotheses on C are equivalent to the condition that C is of even degree and free from multiple components. If C is transversal to a line at infinity $L_\infty = \{x_0 = 0\}$ and $F(x_0 : x_1 : x_2) = 0$ is a homogeneous equation of C, then $M(C)$ can be seen as the possibly singular surface associated to the 2-valued function

$$g := \sqrt{\frac{F}{x_0^{d(F)}}}.$$

since the degree $d(F)$ is even g does not ramifies along $L_\infty - C$.

6.2 Canonical resolution

We describe the *canonical resolution* [BPV] of the surfaces $M(C)$ introduced in §6.1. Let $M^0(C) := M(C)$, $L_0 := L$, $C_0 := C$ and take a singular point $p \in C_0$. Let $\sigma : M' \to M$ be a blowing up at $p_0 := p$, $E_1 := \sigma^{-1}(p_0)$, $\mathcal{O}(E_1)$ the associated line bundle. If the algebraic multiplicity of C_0 at p_0 is ν_0, define the curve $C_1 \subset M'$ and the line bundle L_1 of M' by:

$$C_1 := \sigma^*(C_0) - 2[\frac{\nu_0}{2}]E_1$$

$$L_1 := \sigma^* L \otimes \mathcal{O}(-[\frac{\nu_0}{2}]E_1),$$

where $[\frac{\nu_0}{2}]$ means the greatest integer not exceeding $\frac{\nu_0}{2}$. Since $\mathcal{O}(C_1) = L_1 \otimes L_1$, we can define as above the 2-fold covering $M^1(C)$ over M', which has ramification along C_1.

Let (x, y) be local coordinates of a neighborhood $\mathcal{U}$ of p_0 with $x(p_0) = y(p_0) = 0$ and let (t, y) be local coordinates for $q \in \sigma^{-1}(p_0)$, where $\sigma(t, y) = (ty, y) = (x, y)$. If $\tilde{z}$ is a local coordinate for $L_{1|\mathcal{U}}$, then $M^1(C)$ is given in the local coordinates introduced by

$$\tilde{z}^2 - \tilde{f}(t, y) := \tilde{z}^2 - \frac{f(ty, y)}{y^{2[\frac{\nu_0}{2}]}} = 0, \quad \text{with } C_0 = \{f(x, y) = 0\}.$$

A bimeromorphic map $\phi : M^1(C) \to M^0(C)$ is given in the local coordinates above by

$$\phi((t, y), \tilde{z}) = ((ty, y), y^{[\frac{\nu_0}{2}]}\tilde{z}) = ((x, y), z).$$

After a finite number of blowing ups σ we obtain from C_0 a curve C_l with only ordinary double points. Therefore $C_{l+1} := \sigma^*(C_l) - 2E_{l+1}$ coincides with the strict transform of C_l under a blowing up. Then C_{l+1} is smooth and therefore $M^{l+1}(C)$ is smooth and ramified exactly along C_{l+1} in a neighborhood of E_{l+1}.

Proceeding in this way for all singularities of C_0, we obtain a smooth surface, denoted $\widetilde{M}(C)$, and a bimeromorphic map $\phi : \widetilde{M}(C) \to M(C)$ which is a desingularization of $M(C)$. The map $\pi \circ \phi : \widetilde{M}(C) \to M$ is generically finite of degree two.

If $\pi : M(C) \to M$ is the singular 2-fold covering, $\sigma : M' \to M$ denote the sequence of blowing ups used in the definition of $\widetilde{M}(C)$ and $q : \widetilde{M}(C) \to M'$ is the smooth 2-fold covering obtained, then $\pi \circ \phi = \sigma \circ q$.

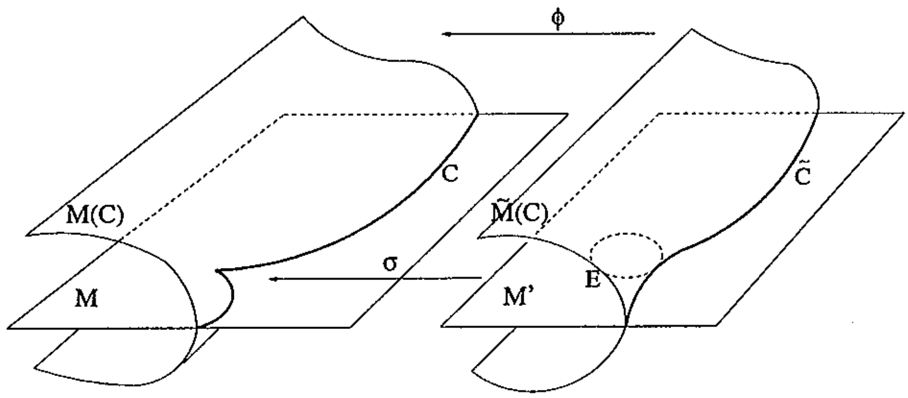

Figure 5: Canonical resolution

Example 6.2.1 Many surfaces are obtained in this way [H]. In the case $M = \mathbf{C}P^2$ and C is a smooth conic, then the smooth surface $M(C)$ is isomorphic to $\mathbf{C}P^1 \times \mathbf{C}P^1$. If C is the union of two lines , then $\widetilde{M}(C)$ is isomorphic to the Hirzebruch surface Σ_2. If C is smooth of degree 4 in $\mathbf{C}P^2$, $M(C)$ is a Del Pezzo surface isomorphic to $\mathbf{C}P^2$ with 7 points blown up. If C has degree 6, $\widetilde{M}(C)$ is a $K3$-surface, etc. Starting with a rational surface we can by this method obtain surfaces of general type in the sense of Enriques-Kodaira classification.

6.3 Iterated 2-fold coverings

The definition of $\widetilde{M}(C)$ in §6.1 and §6.2 can be iterated, in order to obtain ramified coverings of degree 2^n having ramification curves with order of ramification 2.

Let $C^{(1)}$,...,$C^{(n)}$ be n curves in M intersecting transversally, all C_i free from multiple components and verifying $\mathcal{O}(C^{(i)}) = L^{(i)} \otimes L^{(i)}$. We define $\pi_1 : M(C^{(1)}) \to M$ and consider the canonical desingularization $\phi_1 : \widetilde{M}(C^{(1)}) \to M(C^{(1)})$.

Starting now from $\widetilde{M}(C^{(1)})$ and the curve $(\pi_1 \circ \phi_1)^{-1}(C^{(2)}) \subset \widetilde{M}(C^{(1)})$ we define the singular 2-covering

$$\pi_2 : \widetilde{M}(C^{(1)})((\pi_1 \circ \phi_1)^{-1}(C^{(2)})) \to \widetilde{M}(C^{(1)})$$

(the points $C^{(1)} \cap C^{(2)}$ are regular points of $C^{(1)}$ and therefore the curve $(\pi_1 \circ \phi_1)^{-1}(C^{(2)})$ has no multiple components and also verify $(\pi_1 \circ \phi_1)^*\mathcal{O}(C^{(2)}) = (\pi_1 \circ \phi_1)^*(L^{(2)} \otimes L^{(2)})$. The canonical resolution of $\widetilde{M}(C^{(1)})((\pi_1 \circ \phi_1)^{-1}(C^{(2)}))$ will be denoted

$$\phi_2 : \widetilde{M}(C^{(1)}, C^{(2)}) \to \widetilde{M}(C^{(1)})((\pi_1 \circ \phi_1)^{-1}(C^{(2)})).$$

In this way we define $\widetilde{M}(C^{(1)}, ..., C^{(n)})$ and

$$\pi_1 \circ \phi_1 \circ ... \circ \pi_n \circ \phi_n : \widetilde{M}(C^{(1)}, ..., C^{(n)}) \to M$$

is a generically finite map of degree 2^n, with ramification curves with order of ramification 2.

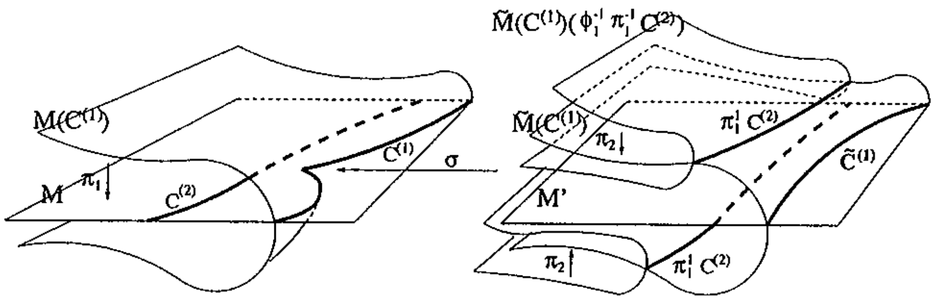

Figure 6: Iterated 2-fold covering

6.4 Pullback of foliations and variation of $\chi(\mathcal{F})$

A *foliation* $\mathcal{F}$ of a neighborhood $\mathcal{U} \subset M$ of an isolated singular point p of a surface M is a regular foliation of $\mathcal{U} - \{p\}$. A local *separatrix* at p is a leaf L of $\mathcal{U} - \{p\}$ having closure given by $L \cup \{p\}$.

Keeping the notation of the previous section, we define starting with $\mathcal{F}$ of M the foliations $\pi_1^*(\mathcal{F})$ of $M(C^{(1)})$ (singular surface) and $\mathcal{G}_1 := \phi_1^*\pi_1^*(\mathcal{F})$ which extends with isolated zeros the foliation

$$(\phi_{1|\widetilde{M}(C^{(1)})-\phi_1^{-1}(p)})^*(\pi_1^*(\mathcal{F}))$$

since $M(C^{(1)})$ is *normal* [C]. In this way we define

$$\mathcal{G} := (\pi_1 \circ \phi_1 \circ \ldots \circ \pi_n \circ \phi_n)^*(\mathcal{F}).$$

Definition 6.4.1 A local non-singular foliation $\mathcal{F}$ has *low contact* with a non $\mathcal{F}$-invariant curve γ at p when the multiplicity of intersection $\nu_p(l,\gamma)$ between the leaf l of $\mathcal{F}$ verifies:

$$\nu_p(l,\gamma) = \begin{cases} \nu_p(\gamma) & \text{if } \nu_p(\gamma) \geq 2 \\ 1 \text{ or } 2 & \text{if } \nu_p(\gamma) = 1. \end{cases}$$

Definition 6.4.2 A foliation $\mathcal{F}$ is in *general position* relative to a curve C at p when i) no local branch at p of C is $\mathcal{F}$-invariant, ii) $p \notin Sing(\mathcal{F})$ and iii) $\mathcal{F}$ has low contact with C at p.

A foliation $\mathcal{F}$ is in general position relative to a curve C when it is in general position relative to C at all points.

Remark 9 Let $\mathcal{F}$ be a foliation in general position relative to a curve C at p with $p = (0,0)$ and $C = \{x^2 + y^{k+1} = 0\}$ (C has a *generalized cusp* at p). Then the order of tangency between $\mathcal{F}$ and C at p, $tang(\mathcal{F},p)$, verifies $tang(\mathcal{F},p) = k+1$.

In fact, let $\mathcal{F}$ be represented by the local vector field

$$X = a(x,y)\frac{\partial}{\partial x} + b(x,y)\frac{\partial}{\partial y}, \quad a(0,0) \neq 0.$$

By definition (cf. §2.4)

$$tang(C,\mathcal{F},p_i) := dim_{\mathbb{C}}\frac{\mathcal{O}_{(\mathbb{C}^2,0)}}{\mathcal{J}},$$

where $\mathcal{J}$ is the ideal

$$\begin{aligned} \mathcal{J} &= < x^2 + y^{k+1}, X(x^2 + y^{k+1}) > \\ &= < x^2 + y^{k+1}, 2a(x,y)x + (k+1)b(x,y)y^k > . \end{aligned}$$

With the holomorphic change of coordinates

$$u = u(x,y) := x + \frac{(k+1)b(x,y)y^k}{2a(x,y)}, \quad v = v(x,y) := y,$$

we obtain for $x = g(u,v)$, $a = a(g(u,v),v)$ and $b = b(g(u,v),v)$:

$$\mathcal{J} =< (u - \frac{(k+1)bv^k}{2a})^2 + v^{k+1}, u >,$$

because $a(0,0) \neq 0$. That is,

$$\begin{aligned} \mathcal{J} &= < u(u - \frac{(k+1)b}{a}v^k) + v^{k+1}(1 + v^{k-1}(\frac{(k+1)b}{2a})^2), u > \\ &= < v^{k+1}, u >, \end{aligned}$$

which proves the assertion.

Remark 10 A germ of surface M in $(\mathbb{C}^3, 0)$ has a *simple singularity of type* $A_{k\geq 1}$ when, in local analytical coordinates (w, x, y), M is represented by

$$A_k : w^2 + x^2 + y^{k+1} = 0,$$

that is, M is a 2-fold covering with $\pi(w, x, y) = (x, y)$, ramifying along $x^2 + y^{k+1} = 0$.

The canonical resolution ϕ of each p of type A_k is well known: $\phi^{-1}(p)$ is composed by a *chain* of k (-2)-curves (having $k-1$ nodal intersections).

Theorem 6.4.3 *Let $C^{(1)}, \ldots, C^{(n)}$ be curves of a surface M with*

i) transversal intersections,

ii) each C_i free from multiple components and verifying $\mathcal{O}(C^{(i)}) = L^{(i)} \otimes L^{(i)}$, for line bundles $L^{(i)}$ on M and

iii) each $C^{(i)}$ with singular points of type $x^2 + y^{k+1} = 0$, for any $k = k(p, i) \geq 1$. Let

$$T : \widetilde{M}(C^{(1)}, \ldots, C^{(n)}) \to M \quad and \quad \mathcal{G} := T^*(\mathcal{F})$$

be the iterated 2-fold covering defined in §6.3 and the pullback of $\mathcal{F}$ of M. If $\mathcal{F}$ is in general position relative to $C^{(1)} \cup \cdots \cup C^{(n)}$, then

$$\chi(\mathcal{G}) = 2^n \chi(\mathcal{F}) - 2^{n-1} c_1(N_{\mathcal{F}}) \cdot \sum_{i=1}^{n} C^{(i)}.$$

In the particular case $M = \mathbf{C}P^2$ and $C^{(i)}$ are curves with even degree $d(C^{(i)})$:

$$\chi(\mathcal{G}) = 2^n \chi(\mathcal{F}) - 2^{n-1} (d(\mathcal{F}) + 2) \sum_{i=1}^{n} d(C^{(i)}).$$

Proof We first prove for $n = 1$. In this case we denote $C^{(1)} = C$, $\pi_1 = \pi$, etc. in the notation introduced for definition of $\widetilde{M}(C)$.

Applying Theorem 3.2.3 to $\mathcal{G} := \phi^*(\pi^*(\mathcal{F}))$ in $\widetilde{M}(C)$

$$\chi(\mathcal{G}) = c_2(\widetilde{M}(C)) - Det(\mathcal{G}) + \sum_{q \in Sing\mathcal{R}(\mathcal{G})} m_q(m_q - 1)$$

and we will compute each term. At first we compute the data of the singularities of $\mathcal{G}$.

By the general position hypothesis on $\mathcal{F}$

$$Det(\mathcal{G}_{|\widetilde{M}(C) - (\pi \circ \phi)^{-1}(C)}) = 2Det(\mathcal{F}).$$

At the points $p \in C - Sing(C)$ both the curve C and the surface $M(C)$ are non-singular and hence ϕ is a local isomorphism. By the general position hypothesis on $\mathcal{F}$, if (u, v) is a local coordinate for $q \in (\pi \circ \phi)^{-1}(p)$

$$(\pi \circ \phi)(u, v) = (u, v^2) = (x, y)$$

with

$$C = \{y = 0\}, \quad \mathcal{F} : d(y - x^l) = 0, \quad l = 1, 2$$

and

$$\mathcal{G} := \phi^* \pi^*(\mathcal{F}) : \; 2v dv - l u^{(l-1)} du = 0.$$

That is, q is either a regular point of $\mathcal{G}$ or a reduced singularity of Morse type (with $Det(\mathcal{G},q) = 1$ and $m_q(m_q-1) = 0$). The second case occurs when $\mathcal{F}$ has a tangency ($l = 2$) with C at $p \in C - Sing$ and the number of such points of tangency is

$$\begin{aligned}\sum_{p\in C-Sing(C)} tang(C,\mathcal{F},p) &= tang(\mathcal{F},C) - \sum_{p\in Sing(C)} tang(\mathcal{F},C,p) \\ &= c_1(N_{\mathcal{F}})\cdot C + C\cdot C + K_M\cdot C - \sum_{p\in Sing(C)} tang(C,\mathcal{F},p),\end{aligned}$$

according to §2.4. By Remark 9 and the hypothesis on the singularities of C

$$tang(C,\mathcal{F},p) = k(p)+1.$$

We conclude

$$\sum_{q\in(\pi\circ\phi)^{-1}(C-Sing(C))} Det(\mathcal{G},q) = c_1(N_{\mathcal{F}})\cdot C + C\cdot C + K_M\cdot C - \sum_{p\in Sing(C)} (k(p)+1).$$

We consider now the singularities of $\mathcal{G}$ along $(\pi\circ\phi)^{-1}(Sing(C))$.

The first remark is that $\pi^*(\mathcal{F})$ in $M(C)$ has exactly two separatrices at $p \in Sing(C) \cap Sing(M(C))$. In fact, in (x,y) local coordinates with $C = \{x^2+y^{k(p)+1} = 0\}$, $M(C)$: $w^2+x^2+y^{k(p)+1} = 0\}$, we can suppose that the leaf l of $\mathcal{F}$ $p = (0,0)$ is $l = \{y = h(x)\}$, with $h(0) = 0$ and $h'(0) = \lambda \in \mathbb{C} - \{\pm\sqrt{-1}\}$. Then for $\pi(w,x,y) = (x,y)$, $\pi^{-1}(l) = \{w^2 + x^2(1+\widetilde{h}(x)) = 0\}$, with $1+\widetilde{h}(0) \neq 0$. With $c(x) := 1+\widetilde{h}(x)$ the two separatrices of $\pi^*(\mathcal{F})$ are $w = \pm\sqrt{-c(x)}x$.

Since there are two separatrices of $\pi^*(\mathcal{F})$ at $p \in Sing(M(C))$ and $\phi_{|\widetilde{M}(C)-\phi^{-1}(p)}$ is an isomorphism onto $M(C)-\{p\}$, then the divisor $\phi^{-1}(p_i)$ is $\mathcal{G}$-invariant.

Assertion: $\mathcal{G}$ has $k(p)+1$ singularities q of Morse type along $\phi^{-1}(p)$.

Proof of the assertion: Keeping the notation of the canonical resolution §6.2, we have $\pi\circ\phi = \sigma\circ q$, where $\sigma : M' \to M$ is a blowing up at p, $q : M^1(C) \to M'$ is the projection ramified along C_1. Let $E_1 = \sigma^{-1}(p)$, in local coordinates given by $\sigma(t,y) = (ty,y) = (x,y)$ and

$$\begin{aligned}C_1 &:= \sigma^*(\{x^2+y^{k(p)+1} = 0\}) - 2[\frac{\nu_p(C)}{2}]E_1 \\ &= \sigma^*(\{x^2+y^{k(p)+1} = 0\}) - 2E_1 \\ &= \{t^2+y^{k(p)-1} = 0\}.\end{aligned}$$

This means that, in local coordinates (t,y), if $k(p) = 1$

$$E_1\cap C_1 = \{(\sqrt{-1},0),(-\sqrt{-1},0)\}$$

and if $k(p)\geq 2$ $E_1\cap C_1 = \{(0,0)\}$. Since $q : M^1(C) \to M'$ ramifies only along C_1, it does not ramify over the point $(t,y) = (\lambda^{-1},0)$ which is the intersection of the strict transform of the leaf l with E_1. Therefore in $M^1(C)$ there are two copies of a singularity isomorphic to $sdz + zds = 0$ (see *Figure* 7 below).

Next steps of the canonical resolution are composed by blowing ups σ at points of $C_1\cap E_1$. Such points are regular points of the strict transform of $\mathcal{F}$ by σ. Therefore $\mathcal{G}$ has along the $\mathcal{G}$-invariant divisor $\phi^{-1}(p)$ a): 2 singularities isomorphic to $sdz+zds = 0$ and b): singularities at the normal crossings of $\phi^{-1}(p)$. But, by Remark 10, $\phi^{-1}(p)$ is composed by a chain of $k(p)$ (-2)-curves with $k(p)-1$ normal intersections and the Assertion is proved.

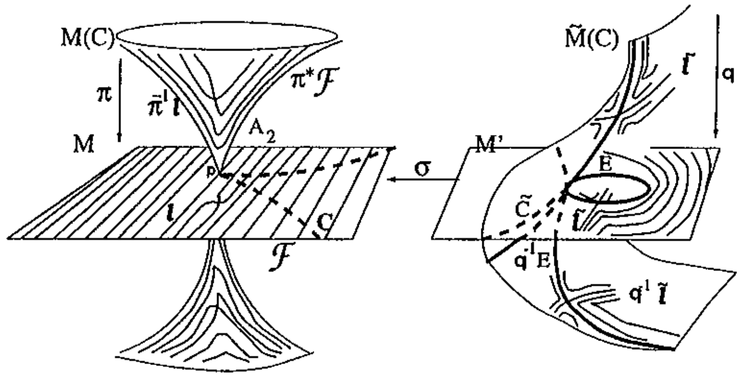

Figure 7: Singularities of $\mathcal{G}$ along $\Phi^{-1}(q) = q^{-1}\sigma^{-1}(p)$

By the hypotheses on the singularities of C [BPV],

$$c_2(\widetilde{M}(C)) = 2c_2(M) + C \cdot C + K_M \cdot C.$$

Then

$$\begin{aligned}
\chi(\mathcal{G}) &= c_2(\widetilde{M}(C)) - Det(\mathcal{G}) + \sum_{q \in Sing\mathcal{R}(\mathcal{G})} m_q(m_q - 1) \\
&= 2(c_2(M) - Det(\mathcal{F}) + \sum_{p \in Sing\mathcal{R}(\mathcal{F})} m_q(m_q - 1)) - c_1(N_{\mathcal{F}}) \cdot C \\
&= 2\chi(\mathcal{F}) - c_1(N_{\mathcal{F}}) \cdot C.
\end{aligned}$$

Now we prove the theorem by induction on n (the number of iterations in the definition of the coverings $\widetilde{M}(C^{(1)}, ..., C^{(n)})$). Supposing the result is true up to $n-1$ iterations

$$\chi(\mathcal{G}_{n-1}) = 2^{n-1}\chi(\mathcal{F}) - 2^{n-2}c_1(N_{\mathcal{F}}) \cdot \sum_{i=1}^{n-1} C^{(i)}$$

and for $\mathcal{G} := \mathcal{G}_n$

$$\chi(\mathcal{G}) = 2\chi(\mathcal{G}_{n-1}) - c_1(N_{\mathcal{G}_{n-1}}) \cdot (\pi_1 \circ \phi_1 \circ ... \circ \pi_{n-1} \circ \phi_{n-1})^{-1}(C^{(n)}).$$

Therefore

$$\begin{aligned}
\chi(\mathcal{G}) &= 2^n\chi(\mathcal{F}) - 2^{n-1}c_1(N_{\mathcal{F}}) \cdot \sum_{i=1}^{n-1} C^{(i)} - \\
&\quad -(\pi_1 \circ \phi_1 \circ ... \circ \pi_{n-1} \circ \phi_{n-1})^* c_1(N_{\mathcal{F}}) \cdot (\pi_1 \circ ... \circ \pi_{n-1} \circ \phi_{n-1})^{-1}(C^{(n)}) \\
&= 2^n\chi(\mathcal{F}) - 2^{n-1}c_1(N_{\mathcal{F}}) \cdot \sum_{i=1}^{n-1} C^{(i)} - 2^{n-1}c_1(N_{\mathcal{F}}) \cdot C^{(n)},
\end{aligned}$$

where in the last equality we use that $(\pi_1 \circ ... \circ \pi_{n-1} \circ \phi_{n-1})$ is generically finite of degree 2^{n-1} and the property

$$c_1(f^*(L)) \cdot f^*(D) = d(f)\, c_1(L) \cdot D.$$

□

7 Appendix A: Genera of fibers

In what follows, we denote by C^2 the self-intersection number $C \cdot C$ of a curve C in a surface M.

Lemma 7.0.4 *An irreducible curve C in M is a (-1)-curve if and only if $C^2 < 0$ and $K_M \cdot C < 0$.*

This Lemma is Proposition III.2.2 of [BPV].

Lemma 7.0.5 *(Zariski, O.)*

Let $M_s = \sum_{i=1}^{m} n_i C_i$ ($n_i > 0$, C_i reduced and irreducible) be a fiber of a connected fibration given by $f : M \to S$. Then

i) $M_s \cdot C_i = 0$ for all i,

ii) if $D := \sum_{i=1}^{m} k_i C_i$, $k_i \in \mathbb{Z}$, then $D^2 \leq 0$,

iii) $D^2 = 0$ in ii) above if and only if $D = rM_s$, with $r \in \mathbb{Q}$ (that is, $pD = qM_s$ with $p \neq 0$, $p, q \in \mathbb{Z}$).

This Lemma is Proposition III.8.2 of [BPV].

We recall that $p_a(C_i) := 1 + \frac{1}{2}(C_i^2 + C_i \cdot K_M)$ (the arithmetical genus) and $g(\widetilde{C_i})$ is the genus of the normalization (geometrical genus).

Definition: The Dual Graph G_{M_s} of the singular fiber M_s is defined by associating a): to each component C_i of M_s a vertex v_i, b): $C_i \cdot C_j$ edges connecting the vertices v_i and v_j and c): $p_a(C_i) - g(\widetilde{C_i})$ loops around the vertex v_i.

We denote the i-th Betti number of the graph G_{M_s} by $b_i(G_{M_s})$, $i = 0, 1$.

Lemma 7.0.6 *(Xiao, G.) Let $M_s = \sum_{i=1}^{m} n_i C_i$ ($n_i > 0$, C_i reduced and irreducible) be a fiber of a connected fibration $f : M \to S$ with $g(M_g) \geq 2$.*

Let $b_{M_s} := b_1(G_{M_s})$ and suppose M_s is minimal. Then

$$g(M_g) \geq b_{M_s} + \sum_{i=1}^{m} g(\widetilde{C_i}),$$

with

$$b_{M_s} = \sum_{i=1}^{m} (p_a(C_i) - g(\widetilde{C_i})) + \frac{\sum_{i=1}^{m} (C_i \cdot \sum_{j \neq i}^{m} C_j)}{2} - (m-1).$$

Moreover, equality holds above if and only if M_s is free from multiple components.

Proof At first, remark that the topological Euler characteristic of the graph G_{M_s} is given by

$$\chi(G_{M_s}) = m - \frac{\sum_{i=1}^{m} (C_i \cdot \sum_{j \neq i}^{m} C_j)}{2} - \sum_{i=1}^{m} (p_a(C_i) - g(\widetilde{C_i})),$$

and therefore

$$\begin{aligned} b_{M_s} := b_1(G_{M_s}) &= b_0(G_{M_s}) - \chi(G_{M_s}) \\ &= 1 - \chi(G_{M_s}), \end{aligned}$$

that is,

$$b_{M_s} = \sum_{i=1}^{m} (p_a(C_i) - g(\widetilde{C_i})) + \frac{\sum_{i=1}^{m} (C_i \cdot \sum_{j \neq i}^{m} C_j)}{2} - (m-1),$$

as asserted.

We also remark that, by Zariski Lemma 7.0.5,

$$(M_s)^2_{red} := (\sum_{i=1}^{m} C_i)\cdot(\sum_{i=1}^{m} C_i) \leq 0,$$

that is,

$$\text{(i)} \quad -\sum_{i=1}^{m} C_i^2 \geq \sum_{i=1}^{m}(C_i \cdot \sum_{j\neq i}^{m} C_j).$$

Also

$$\begin{aligned} 2g(M_g)-2 = 2p_a(M_g)-2 &= 2p_a(M_s)-2 \\ &= M_s^2 + M_s\cdot K_M \\ &= M_s \cdot K_M. \end{aligned}$$

We assert that

$$\text{(ii)} \quad M_s\cdot K_M \geq (M_s)_{red}\cdot K_M := \sum_{i=1}^{m} C_i \cdot K_M.$$

To prove *(ii)*, it is enough to prove that $\forall i\ C_i\cdot K_M \geq 0$. Suppose by absurd, $C_i \cdot K_M < 0$. Then if $C_i^2 < 0$ we conclude, by Lemma 7.0.4, that C_i is a (-1)-curve, contradicting the minimality of M_s. If $C_i^2 = 0$, then by Zariski Lemma, $C_i = \frac{1}{N}M_s$. Hence, by our assumption $\frac{M_s}{N}\cdot K_M < 0$, and then $M_s \cdot K_M < 0$, which gives the contradiction:

$$0 < 2g(M_g) - 2 = M_s \cdot K_M < 0,$$

which proves *(ii)*.

Collecting the information above and using *(i)* and *(ii)*, we get:

$$\begin{aligned} 2g(M_g)-2 = M_s\cdot K_M &\geq (M_s)_{red}\cdot K_M = \sum_{i=1}^{m} C_i\cdot K_M \\ &= \sum_{i=1}^{m}(2p_a(C_i) - C_i^2) - 2m \\ &\geq \sum_{i=1}^{m}(2p_a(C_i) + C_i \cdot \sum_{j\neq i} C_j) - 2m \\ &= \sum_{i=1}^{m} 2g(\widetilde{C_i}) + 2b_{M_s} - 2, \end{aligned}$$

that is,

$$g(M_g) \geq b_{M_s} + \sum_{i=1}^{m} g(\widetilde{C_i}).$$

At last, remark that the equality in *(i)* above means by Zariski Lemma $(M_s)_{red} = \frac{1}{N}M_s$ with $N \geq 1$. If also there is equality in *(ii)*, then

$$\frac{1}{N}M_s\cdot K_M = (M_s)_{red}\cdot K_M = M_s\cdot K_M = 2g(M_g) - 2 > 0,$$

which implies $N = 1$ and $M_s = (M_s)_{red}$.

□

8 Appendix B: Formulas for pencils

We give new proofs of some results found in [P] and [Pa] relating the genus of a generic curve C of a pencil $\mathcal{F}$ in $\mathbf{C}P^2$ with data of the pencil ($d(\mathcal{F})$ and informationa about its singularities).

Proposition 8.0.7 *Suppose $\mathcal{F}$ is a pencil in $\mathbf{C}P^2$ with irreducible generic curve. Let $\mathcal{R}(\mathcal{F})$ be a resolution such that the reduced associated foliation is the fibration $\widetilde{\mathcal{F}}$. If C is a generic curve of $\mathcal{F}$, then*

$$(d(\mathcal{F})+2)d(C) = \sum_{p \in Sing\mathcal{R}(\mathcal{F})} m_p \nu_p$$

and

$$g(C) = 1 + \frac{1}{2} \sum_{p \in Sing\mathcal{R}(\mathcal{F})} \left(1 - \frac{3m_p}{d(\mathcal{F})+2}\right)\nu_p(C).$$

where ν_p is the algebraic multiplicity of C and its strict transforms.

In particular, if the singularities of $\mathcal{F} \cap C$ are of radial type, then $d(\mathcal{F}) > 4$ implies $g(C) \geq 2$. The pencil in $\mathbf{C}P^2$ generated by two smooth cubics with transversal intersections has $d(\mathcal{F}) = 4$ and $g(C) = 1$.

Also by the formula, if $d(\mathcal{F}) + 2 > 3m_p$ for all $p \in Sing\mathcal{R}(\mathcal{F})$ then $g(C) \geq 2$.

Proof After a resolution $\mathcal{R}(\mathcal{F})$, the strict transform $\widetilde{C}$ of C is a generic fiber of the fibration $\widetilde{\mathcal{F}}$ and then $\widetilde{C} \cdot \widetilde{C} = 0$. By the formula of sum of Poincaré-Hopf indices of §2.4

$$c_1(N_{\widetilde{\mathcal{F}}}) \cdot \widetilde{C} - Z(\widetilde{C}, \widetilde{\mathcal{F}}) = \widetilde{C} \cdot \widetilde{C}.$$

and therefore

$$c_1(N_{\widetilde{\mathcal{F}}}) \cdot \widetilde{C} = 0.$$

Denoting $\sigma : \widetilde{M} \to \mathbf{C}P^2$ the sequence of blowing ups of $\mathcal{R}(\mathcal{F})$, we have

$$c_1(N_{\widetilde{\mathcal{F}}}) \cdot \widetilde{C} = (\sigma^* c_1(N_{\mathcal{F}}) - \sum_{p \in Sing\mathcal{R}(\mathcal{F}) - Sing\widetilde{\mathcal{F}}} m_p E_p) \cdot (\sigma^*(C) - \sum_{p \in Sing\mathcal{R}(\mathcal{F}) - Sing\widetilde{\mathcal{F}}} \nu_p E_p),$$

where the sums $\sum_p m_p E_p$ and $\sum_p \nu_p E_p$ are relative to the strict transforms of $\mathcal{F}$ and C, respectively, along $\mathcal{R}(\mathcal{F})$. That is,

$$c_1(N_{\mathcal{F}}) \cdot C - \sum_{p \in Sing\mathcal{R}(\mathcal{F})} m_p \nu_p = 0.$$

Which gives in $\mathbf{C}P^2$

$$(d(\mathcal{F})+2)d(C) - \sum_{p \in Sing\mathcal{R}(\mathcal{F})} m_p \nu_p = 0.$$

To prove the second assertion, we write

$$d(C) = \frac{\sum_{p \in Sing\mathcal{R}(\mathcal{F})} m_p \nu_p}{d(\mathcal{F})+2}$$

and we recall that for a irreducible generic curve C of $\mathcal{F}$

$$d(C) = \frac{1}{3}(2 - 2g(C) + \sum_{p \in Sing\mathcal{R}(\mathcal{F})} \nu_p),$$

according to Remark 4 of §4.3. This gives the result

$$g(C) = 1 + \frac{1}{2} \sum_{p \in Sing\mathcal{R}(\mathcal{F})} (1 - \frac{3m_p}{d(\mathcal{F})+2})\nu_p.$$

□

Corollary 8.0.8 *(of Proposition 8.0.7) Suppose $\mathcal{F}$ is a pencil in $\mathbf{C}P^2$ with irreducible generic curve. Suppose that all singularities of $\mathcal{F}$ with local meromorphic first integral are dicritical points without singularities along the exceptional line of a blowing up.*

If C is a generic curve, then

$$2(g(C) - 1 + d(C)) = (d(\mathcal{F}) + 1)d(C) - \sum_{p \in Sing(\mathcal{F})} \nu_p(\mathcal{F})\nu_p(C),$$

or equivalently

$$g(C) = 1 + \frac{1}{2} \sum_{p \in Sing(\mathcal{F})} (1 - \frac{3m_p(\mathcal{F})}{d(\mathcal{F})+2})\nu_p(C).$$

This corollary is found in [Pa] in the case where the singularities with meromorphic first integral are of radial type.

Proof A dicritical singularity has $m_p(\mathcal{F}) = \nu_p(\mathcal{F}) + 1$, where $\nu_p(\mathcal{F})$ is the algebraic multiplicity of $\mathcal{F}$. By Proposition 8.0.7

$$(d(\mathcal{F}) + 1)d(C) - \sum_{p \in Sing(\mathcal{F})} \nu_p(\mathcal{F})\nu_p(C) + d(C) - \sum_{p \in Sing(\mathcal{F})} \nu_p(C) =$$

$$= (d(\mathcal{F}) + 2)d(C) - \sum_{p \in Sing(\mathcal{F})} m_p(\mathcal{F})\nu_p(C) = 0.$$

Since

$$d(C) = \frac{1}{3}(2 - 2g(C) + \sum_{p \in Sing\mathcal{R}(\mathcal{F})} \nu_p),$$

we obtain

$$d(C) - \sum_{p \in Sing\mathcal{R}(\mathcal{F})} \nu_p = -2(g(C) - 1 + d(C))$$

and we conclude that

$$(d(\mathcal{F}) + 1)d(C) - \sum_{p \in Sing(\mathcal{F})} \nu_p(\mathcal{F})\nu_p(C) = 2(g(C) - 1 + d(C)).$$

□

Corollary 8.0.9 *(of Proposition 8.0.7)*

Let $\mathcal{F}$ be a pencil of $\mathbf{C}P^2$ with irreducible generic curve. Suppose that all the points p with local meromorphic first integral are given in local coordinates (x, y) by:

$$\omega = \lambda_p x dy - y dx + \omega_2, \quad \lambda_p \in \mathbb{N},$$

where the 1-form ω_2 vanishes with order at least two. If C denotes a generic curve of $\mathcal{F}$, then

$$2(g(C) - 1 + d(C)) = (d(\mathcal{F}) + 1)d(C) - \sum_{p \in Sing(\mathcal{F})} \nu_p(C),$$

or equivalently

$$g(C) = 1 + \frac{1}{2} \sum_{p \in Sing(\mathcal{F})} [(\lambda_p + 1)(1 - \frac{3}{d(\mathcal{F}) + 2}) - 1]\nu_p(C).$$

Proof At a singularity $\omega = \lambda_p x dy - y dx + \omega_2$, C has smooth local branches of type $\{x^{\lambda_p} - c_i y + h.o.t = 0\}$ with $c_i \neq 0$ and $i = 1, ..., \nu_p(C)$. The elimination of such base-points amounts to λ_p blowing ups (the λ_p-th is done at a radial point), which gives

$$\sum_{p \in Sing\mathcal{R}(\mathcal{F})} m_p \nu_p = \sum_{p \in Sing(\mathcal{F})} (\lambda_p + 1)\nu_p(C).$$

Since

$$\sum_{p \in Sing\mathcal{R}(\mathcal{F})} \nu_p = \sum_{p \in Sing(\mathcal{F})} \lambda_p \nu_p(C),$$

$$d(C) - \sum_{p \in Sing(\mathcal{F})} \lambda_p \nu_p(C) = -2(g(C) - 1 + d(C)).$$

Then

$$(d(\mathcal{F})+2)d(C) - \sum_{p \in Sing\mathcal{R}(\mathcal{F})} m_p \nu_p = (d(\mathcal{F})+1)d(C) - \sum_{p \in Sing(\mathcal{F})} \nu_p(C) + d(C) - \sum_{p \in Sing(\mathcal{F})} \lambda_p \nu_p(C)$$

$$= (d(\mathcal{F}) + 1)d(C) - \sum_{p \in Sing(\mathcal{F})} \nu_p(C) - 2(g(C) - 1 + d(C)) = 0.$$

The second assertion is immediate using

$$d(C) = \frac{\sum_{p \in Sing(\mathcal{F})} (\lambda_p + 1)\nu_p(C)}{d(\mathcal{F}) + 2}.$$

□

Example 8.0.10 Suppose $\mathcal{F}$ is a pencil in $\mathbf{C}P^2$ with irreducible generic curve and singularities with local meromorphic first integral of type: $\omega = \lambda_p x dy - y dx + \omega_2$, $\lambda_p \in \mathbb{N}$. Then as proved

$$g(C) = 1 + \frac{1}{2} \sum_{p \in Sing(\mathcal{F})} [(\lambda_p + 1)(1 - \frac{3}{d(\mathcal{F}) + 2}) - 1]\nu_p(C)$$

The formula above implies that if $d(\mathcal{F}) > 4$ then $g(C) \geq 2$. Moreover, if $d(\mathcal{F}) > 4$ and $g(C) \geq 2$ are given, then the formula implies that there is an upper bound to

$$\sum_{p \in Sing(\mathcal{F})} (\lambda_p + 1)\nu_p(C)$$

and hence that there is an upper bound to $d(C)$, because as already remarked

$$d(C) = \frac{1}{3}(2 - 2g(C) + \sum_{p \in Sing(\mathcal{F})} \lambda_p \nu_p(C)).$$

If $d(\mathcal{F}) = 4$ and the singularities with meromorphic first integral are of radial type ($\lambda_p = 1$), the formula implies $g(C) = 1$. If $d(\mathcal{F}) = 2$ and the singularities with meromorphic first integral are of type $\omega = 3xdy - ydx + \omega_2$, then also $g(C) = 1$ (these two examples are realized by the elliptic pencil $\mathcal{F}_{I_3}$ of in Example 4.2.6 and the elliptic pencil $\mathcal{F}_3$ of Example 5.1.2). In these two cases the formula gives no upper bound to $\sum_{p \in Sing(\mathcal{F})} \nu_p(C)$ and therefore we cannot assert that there is an upper bound to $d(C) = \frac{1}{3} \sum_{p \in Sing(\mathcal{F})} \nu_p(C)$.

If $d(\mathcal{F}) = 3$ and the singularities with local meromorphic first integral are of radial type ($\lambda_p = 1$), the formula implies $g(C) \leq 0$ and since C is irreducible $g(C) = 0$. Hence $\sum_{p \in Sing(\mathcal{F})} \nu_p(C) = 10$ and $d(C) = 4$.

If $d(\mathcal{F}) = 2$ and the singularities with local meromorphic first integral are of radial type ($\lambda_p = 1$), the formula implies $g(C) = 0$, $\sum_{p \in Sing(\mathcal{F})} \nu_p(C) = 4$ and $d(C) = 2$. These conditions are realized by the pencil generated by two smooth conics with transversal intersections.

If $d(\mathcal{F}) = 2$ and for all singularities with local meromorphic first integral $\lambda_p = 2$, then the formula implies $g(C) = 0$, $\sum_{p \in Sing(\mathcal{F})} \nu_p(C) = 8$ and $d(C) = 6$.

If $d(\mathcal{F}) = 1$ and for all singularities with local meromorphic first integral $\lambda_p = 2$, then $g(C) = 0$, $\sum_{p \in Sing(\mathcal{F})} \nu_p(C) = 2$ and $d(C) = 2$. It is realized by the pencil generated by a smooth conic S and a double line $2L$ with $S \cap L$ transversal.

Example 8.0.11 Consider homogeneous coordinates $(x_0 : x_1 : x_2)$ in $\mathbf{C}P^2$ and the pencils $\mathcal{F}_p$ in $\mathbf{C}P^2$ extending the foliation given in affine coordinates $(x, y) = (\frac{x_1}{x_0}, \frac{x_2}{x_0})$ by $\omega = pxdy + ydx$, $p \in \mathbb{N}$. Then $d(\mathcal{F}_p) = 1$ and $\mathcal{F}_p$ has rational first integral

$$\psi(x_0 : x_1 : x_2) = \frac{x_1^p x_2}{x_0^{p+1}},$$

that is, $\mathcal{F}_p$ is composed by generic curves C with $g(C) = 0$ (because C has a rational parameterization) and $d(C) = p + 1$. Remark that the critical curves of $\mathcal{F}_p$ are given by $C_1 := pL_1 + L_2$ and $C_2 := (p+1)L_0$ where $L_i = \{x_i = 0\}$.

We assert that the generic curve C of $\mathcal{F}_p$ has

a) only one local branch at $z := L_1 \cap L_0$, which is an *ordinary cusp* of order p given by $\{x^p - y^{p+1} = 0\}$ and

b) at $w := L_2 \cap L_0$ C has an unique smooth branch.

We remark that the assertion agrees with the genus formula:

$$2g(C) = 0 = (p + 1 - 1)(p + 1 - 2) - p(p - 1).$$

To prove the assertion, remark that in local coordinates around z, the foliation is given by

$$\omega = (p + 1)vdu - pudv + \omega_2$$

and the local separatrices are generically of type $v^{p+1} - tu^p + h.o.t = 0$, $t \in \mathbb{C}^*$, that is an ordinary cusp of order p. By a blowing up the strict transform of each separatrix is smooth of type $x = t^p$, where $t = 0$ is a local equation of the exceptional line. The foliation is represented around w by

$$\omega = (p + 1)vdu - udv + \omega_2$$

and the separatrices are generically of local type $v^{p+1} - tu = 0$, $t \in \mathbb{C}^*$. By Proposition 8.0.7,

$$3(p+1) - \sum_{q \in Sing\mathcal{R}(\mathcal{F},z)} m_q\nu_q - \sum_{q \in Sing\mathcal{R}(\mathcal{F},w)} m_q\nu_q = (d(\mathcal{F}_p) + 2)d(C) - \sum_{p \in Sing\mathcal{R}(\mathcal{F})} m_p\nu_p = 0$$

and denoting $n_z(C)$ and $n_w(C)$ respectively the number of local branches of C at z and w we obtain

$$3(p + 1) - ((p + 1)n_z(C) + pn_z(C)) - ((p + 2)n_w(C)) = 0,$$

that is $n_z(C) = n_w(C) = 1$.

Remark 11 We give a *geometric* interpretation for the formula of Corollary 8.0.9.

Let $\phi : \widetilde{C} \to C \subset \mathbf{C}P^2$ be a normalization of C. Take a point $q \in \mathbf{C}P^2 - C$ such that

i) the order of contact between any line $L = p.q$ and C verifies $\nu_p(L, C) = \nu_p(C)$, if $\nu_p(C) > 1$ and

ii) if p is a regular point of C and $L = p.q = T_p(C)$, then $\sum_{p \in C} \nu(L, C) = 2$.

Under these hypotheses, consider the set $\mathcal{P} \cong \mathbf{C}P^1$ of lines passing by q and define:

$$\pi_q : \widetilde{C} \to \mathcal{P}$$

$$\pi_q(x) := \phi(x).q$$

where $\phi(x).q$ is a line. The degree of the map π_q is $d(C)$. At a point $x \in \widetilde{C}$ we can write, in local analytic coordinate z with $z(x) = 0$, $\pi_q(z) = z^{n_x} f(z)$, with $f(0) \neq 0$. Defining $b_x := n_x - 1$, we obtain by Riemann-Hurwitz applied to π_q:

$$2(g(C) - 1) := 2(g(\widetilde{C}) - 1) = -2d(C) + \sum_{x \in \widetilde{C}} b_x.$$

If $x \in \widetilde{C}$ is such that $\phi(x) \in Sing(C)$ and γ is the local irreducible branch of C at $\phi(x)$ such that $x = \phi^{-1}(\gamma)$, then $b_x = \nu_{\phi(x)}(\gamma) - 1$ (by the definition of $\nu(\gamma)$ and the choice of q). By the hypotheses on the singularities, $\gamma = \{ty^{\lambda_p} - x = 0\}$,$t \in \mathbb{C}^*$ is smooth and then $b_x = 0$.

Then the points with $b_x \geq 1$ (ramification points for π_q) have regular images $\phi(x) \in C$. Since ϕ is a local isomorphism, the ramification of π_q is due to tangencies between $L = \phi(x).q$ and C at p. We conclude that, for $L := \phi(x).q$,

$$\sum_{x \in \widetilde{C}} b_x = \sum_{\phi(x) \in C - Sing(C)} (\nu_{\phi(x)}(L, C) - 1).$$

Hence, for $L := p.q$,

$$2(g(C) - 1) + d(C)) = \sum_{p \in C - Sing(C)} (\nu_p(L, C) - 1).$$

Let (x, y) be affine coordinates of $\mathbf{C}P^2 - L_\infty$ such that $L_\infty \cap Sing(\mathcal{F}) = \emptyset$. Suppose that the differential equation representing $\mathcal{F}$ is

$$\frac{dy}{dx} = \frac{P(x,y)}{Q(x,y)} \qquad \gcd(P, Q) = 1.$$

Consider now a fixed value $\frac{dy}{dx} = \theta \in \mathbb{C}$ such that the family of parallel affine lines

$$L_\mu := \mathbb{C}(1, \theta) + \mu \quad \mu \in \mathbb{C}^2$$

contains a point $q \in L_\infty$ satisfying conditions i) and ii) for the definition of π_q relatively to the generic curve C of $\mathcal{F}$.

We consider now the map π_q relative to such $q \in L_\infty$. Consider the projective curve given in affine coordinates by

$$S_\theta = \{P(x, y) - \theta Q(x, y) = 0\}.$$

Then $d(S_\theta) = d(\mathcal{F}) + 1$ and the (isolated) singularities of S_θ have algebraic multiplicity $\nu_p(S_\theta) = \nu_p(\mathcal{F})$ (by definition). The intersection $S_\theta \cap C$ is composed by a): regular points of $\mathcal{F}$ at C with tangent $T_p(C) = L_\mu$ for some μ or b): singular points of $\mathcal{F}$ at C (see *Figure* 8 below).

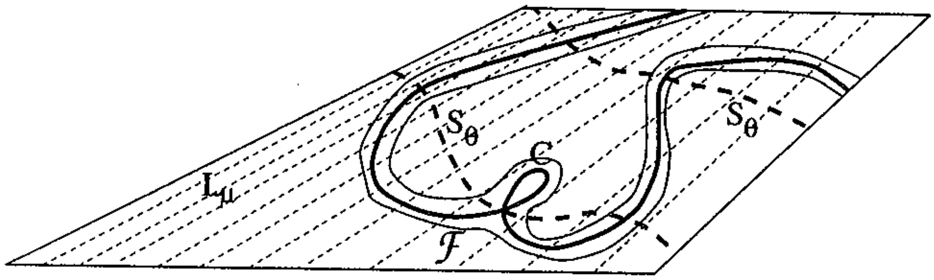

Figure 8: The curve of tangencies S_θ

Hence, as already proved,

$$2(g(C)-1+d(C)) = \sum_{p\in C-Sing(C)} (\nu_p(L_\mu, C)-1)$$

and since C is generic, $Sing(\mathcal{F})\cap C = Sing(C)$; that is,

$$\begin{aligned} 2(g(C)-1+d(C)) &= \sum_{p\notin Sing(S_\theta)} \nu_p(S_\theta, C) \\ &= S_\theta\cdot C - \sum_{p\in Sing(S_\theta)} \nu_p(S_\theta, C) \\ &= (d(\mathcal{F})+1)d(C) - \sum_{p\in Sing(S_\theta)} \nu_p(S_\theta, C). \end{aligned}$$

Since C has at p $\nu_p(C)$ local branchs $\gamma = \{ty^{\lambda_p} - x = 0\}$ and the tangent at p of S_θ is given by $\{y - px = 0\}$, we obtain $\nu_p(S_\theta, C) = \nu_p(S_\theta)\nu_p(C) = \nu_p(C)$ and the formula

$$2(g(C)-1+d(C)) = (d(\mathcal{F})+1)d(C) - \sum_{p\in Sing(\mathcal{F})} \nu_p(C),$$

as desired.

Recall the *projective duality* D which associates to each projective line a point:

$$D(\{ax_0+bx_1+cx_2=0\}) = (a:b:c)\in \mathbf{C}P^2.$$

Restrict to the tangent lines of a plane curve C the map D is well-defined at singular points of C and, if C is not a line, $C' = D(C)$ is a curve called the *dual curve* of C. As it is known [GH], the degree $d(C')$ of the dual curve C' of a curve C having *smooth* local irreducible branchs is $d(C') = 2(g(C)-1+d(C))$. Hence what was proved above is that

$$d(C') = (d(\mathcal{F})+1)d(C) - \sum_{p\in Sing\mathcal{R}(\mathcal{F})} \nu_p(C).$$

This fact is a cornerstone of the proof of Corollary 8.0.9 (and generalizations) found in [Pa]. In [GR] there are generalizations of these geometric ideas.

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