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**Hilbert modular foliations on the projective plane.** (English)

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It is known that the topology of one-dimensional (singular) holomorphic foliations by analytic curves is quite complicated in general. For example, all the phase curves of a generic complex polynomial vector field are dense. There is an old open conjecture (going back to D. V. Anosov) saying that generically all but a countable number of leaves are simply connected (see the paper [*Y. Ilyashenko*, Selected topics in differential equations with real and complex time. Normal forms, bifurcations and finiteness problems in differential equations, 317–354, NATO Sci. Ser. II Math. Phys. Chem., 137, Kluwer Acad. Publ., Dordrecht, (2004)] and its bibliography).

The paper under review studies an important class of holomorphic foliations on $\mathbb{C}P^2$ with isolated singularities that are called *Hilbert modular foliations*, see e.g., [*M. Brunella*, Birational geometry of foliations. (English) Publicaoes Matemáticas do IMPA. Rio de Janeiro: Instituto Nacional de Matemática Pura e Aplicada (IMPA). iv, 138 p. (2004; Zbl 1082.32022)]. Their characterization in terms of the Kodaira dimension and nefness of the cotangent bundle was recently obtained in [*M. Brunella*, Invent. Math. 152, No. 1, 119–148 (2003; Zbl 1029.32014)]. Hilbert modular foliations are constructed in a purely geometric way as follows. Take an appropriate discrete subgroup of $\Gamma \subset PSL_2(\mathbb{R}) \times PSL_2(\mathbb{R})$ of an "arithmetic" origin. It acts on $\mathbb{H}^2 = \mathbb{H} \times \mathbb{H}$. Take the (compactified) quotient and its minimal desingularization. The latter is a nonsingular surface called *a modular surface*. It carries a holomorphic singular foliation that is a "compactification" of the quotient of the horizontal (or vertical) foliation of $\mathbb{H}^2$ by Lobatchevsky (Poincaré) planes. Any holomorphic foliation on $\mathbb{C}P^2$ with isolated singularities is defined by a polynomial vector field in each affine chart. It appears that some examples of modular surfaces are birationally equivalent to the complex projective plane (see [*F. Hirzebruch*, Russ. Math. Surv. 31 (5), 96–110 (1976; Zbl 0356.14010)], [*F. Hirzebruch*, Modular Funct. of one Var. VI, Proc. int. Conf., Bonn 1976, Lect. Notes Math. 627, 287–323 (1977; Zbl 0369.10017)]). Thus, they are obtained by blowing-ups of appropriate polynomial vector fields. Four of these foliations are studied in the paper under review (Theorems 2 and 4). The authors obtain explicit formulas for the corresponding vector fields. To obtain the formulas, the authors show that there is a configuration of complex lines passing through the singularities so that each line contains "many" singularities. Comparison of their number on each line with the degree of the foliation implies invariance of each line. This together with some computer algebra yields the formulas for the corresponding vector fields. Theorem 1 concerns general properties of *reduced* modular foliations, i.e. obtained after reducing the singularities in Seidenberg's sense, see the previously cited lecture notes by M. Brunella. It says that the algebraic invariant curves are rational, all the nonalgebraic leaves are dense, all but a finite number of them are conformally equivalent to the unit disc, the foliation is transversally projective outside the divisor supported on the invariant algebraic curves and the cotangent bundle of the foliation is a unique bundle in its Picard class that is a cotangent bundle of some holomorphic foliation.

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37F75 Holomorphic foliations and vector fields
14G35 Modular and Shimura varieties
11F41 Hilbert modular forms and surfaces